HILBERT AND CHOW SCHEMES OF POINTS, SYMMETRIC PRODUCTS AND DIVIDED POWERS

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Abstract. Let X be a quasi-projective S-scheme. We explain the relation between the Hilbert scheme of d points on X, the d^{th} symmetric product of X, the scheme of divided powers of X of degree d and the Chow scheme of zero-cycles of degree d on X with respect to a given projective embedding $X \hookrightarrow \mathbb{P}(\mathcal{E})$. The last three schemes are shown to be universally homeomorphic with isomorphic residue fields and isomorphic in characteristic zero or outside the degeneracy loci. In arbitrary characteristic, the Chow scheme coincides with the scheme of divided powers for a sufficiently ample projective embedding.

Introduction

Let X be a quasi-projective S-scheme. The purpose of this article is to explain the relation between

a) The Hilbert scheme of points $\mathrm{Hilb}^d\big(X/S\big)$ parameterizing zero-dimensional subschemes of X of degree d.
b) The d^{th} symmetric product $\mathrm{Sym}^d(X/S)$.
c) The scheme of divided powers $\Gamma^d(X/S)$ of degree d.
d) The Chow scheme $\mathrm{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}(\mathcal{E})\big)$ parameterizing zero dimensional cycles of degree d on X with a given projective embedding $X \hookrightarrow \mathbb{P}(\mathcal{E})$.

If X/S is not quasi-projective then none of these objects need exist as schemes but the first three do exist in the category of algebraic spaces separated over S [Ryd08, Ryd07b, **I**]. Classically, only the Chow *variety* of $X \hookrightarrow \mathbb{P}(\mathcal{E})$ is defined but we will show that for zero-cycles there is a natural Chow *scheme* whose underlying reduced scheme is the Chow variety.

There are canonical morphisms

$$\mathrm{Hilb}^d(X/S) \to \mathrm{Sym}^d(X/S) \to \Gamma^d(X/S) \to \mathrm{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}(\mathcal{E}^k)\big)$$

where $k \geq 1$ and $X \hookrightarrow \mathbb{P}(\mathcal{E}^k)$ is the Veronese embedding. The last two of these are universal homeomorphisms with trivial residue field extensions and are isomorphisms if S is a $\mathbb{Q}$-scheme. If S is arbitrary and X/S is flat then the second morphism is an isomorphism. For arbitrary X/S the third morphism is not an

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isomorphism. In fact, the Chow scheme may depend on the chosen embedding as shown by Nagata [Nag55]. However, we will show that the third morphism is an isomorphism for *sufficiently large k*. Finally, we show that all three morphisms are isomorphisms outside the degeneracy locus.

The comparison between the last three schemes uses weighted projective structures. Given a projective embedding $X \hookrightarrow \mathbb{P}(\mathcal{E})$, there is an induced weighted projective structure on the symmetric product $\mathrm{Sym}^d(X)$. This follows from standard invariant theory, using the Segre embedding $(X/S)^d \hookrightarrow \mathbb{P}(\mathcal{E}^{\otimes d})$. In characteristic zero, this weighted projective structure on $\mathrm{Sym}^d(X)$ is actually projective, i.e., all generators have degree one. In positive characteristic, the weighted projective structure is "almost projective": the sheaf $\mathcal{O}(1)$ on $\mathrm{Sym}^d(X)$ is ample and generated by its global sections. This is a remarkable fact as for a general quotient, the sheaf $\mathcal{O}(1)$ of the weighted projective structure is usually not even invertible.

Since $\mathcal{O}(1)$ is generated by its global sections, we obtain a projective morphism $\mathrm{Sym}^d(X) \to \mathbb{P}(\mathrm{TS}^d_{\mathcal{O}_S}(\mathcal{E}))$. In characteristic p, the ample sheaf $\mathcal{O}(1)$ is not always very ample, and thus this morphism is usually not a closed immersion. It is however a *universal homeomorphism* onto its image — the Chow variety. In summary, the reason that the Chow variety depends on the choice of projective embedding is that it is the image of an invariant object, the symmetric product, under a projective morphism induced by an ample sheaf which is not always very ample.

Given a projective embedding $X \hookrightarrow \mathbb{P}(\mathcal{E})$, the scheme of divided powers $\Gamma^d(X)$ has a similar weighted projective structure with similar properties. We define the Chow *scheme* to be the image of the analogous morphism $\Gamma^d(X) \to \mathbb{P}(\Gamma^d_{\mathcal{O}_S}(\mathcal{E}))$. For a sufficiently ample projective embedding, e.g., take the Veronese embedding $X \hookrightarrow \mathbb{P}(\mathrm{S}^k(\mathcal{E}))$ for a sufficiently large k, this morphism is a closed immersion. In particular, it follows that $\Gamma^d(X)$ is projective.

Contents

1. The algebra of divided powers and symmetric tensors

We begin this section by briefly recalling the definition of the algebra of divided powers $\Gamma_A(M)$ and the multiplicative structure of $\Gamma^d_A(B)$. We then give a sufficient and necessary condition for $\Gamma^d_A(M)$ to be generated by $\gamma^d(M)$. This generalizes the sufficiency condition given by Ferrand [Fer98, Lem. 2.3.1]. The condition is essentially that every residue field of A should have at least d elements. Similar conditions on the residue fields reappear in Sections 3.1 and 5.2. Finally, we recall some explicit degree bounds on the generators of $\Gamma^d_A(A[x_1, x_2, \dots, x_r])$.

1.1. **Divided powers and symmetric tensors.** This section is a quick review of the results needed from [Rob63, Rob80]. Also see [Fer98, **I**].

Notation (1.1.1). Let A be a ring and M an A-algebra. We denote the d^{th} tensor product of M over A by $\mathrm{T}^d_A(M)$. We have an action of the symmetric group $\mathfrak{S}_d$ on $\mathrm{T}^d_A(M)$ permuting the factors. The invariant ring of this action is the algebra of symmetric tensors which we denote by $\mathrm{TS}^d_A(M)$. By $\mathrm{T}_A(M)$ and $\mathrm{TS}_A(M)$ we denote the graded A-modules $\bigoplus_{d\geq 0} \mathrm{T}^d_A(M)$ and $\bigoplus_{d\geq 0} \mathrm{TS}^d_A(M)$ respectively.

(1.1.2) Let A be a ring and let M be an A-module. Then there exists a graded A-algebra, the algebra of divided powers, denoted $\Gamma_A(M) = \bigoplus_{d\geq 0} \Gamma^d_A(M)$ equipped with maps $\gamma^d : M \to \Gamma^d_A(M)$ such that, denoting the multiplication with $\times$ as in [Fer98], we have that for every $x, y \in M$, $a \in A$ and $d, e \in \mathbb{N}$

$$\Gamma^0_A(M) = A,\ \gamma^0(x) = 1,\ \Gamma^1_A(M) = M,\ \text{and } \gamma^1(x) = x \tag{1.1.2.1}$$

$$\gamma^d(ax) = a^d\gamma^d(x)$$

$$\gamma^d(x+y) = \textstyle\sum_{d_1+d_2=d} \gamma^{d_1}(x) \times \gamma^{d_2}(y)$$

$$\gamma^d(x) \times \gamma^e(x) = \binom{d+e}{d}\gamma^{d+e}(x)$$

Using (1.1.2.1) we identify A with $\Gamma^0_A(M)$ and M with $\Gamma^1_A(M)$. If $(x_\alpha)_{\alpha\in\mathcal{I}}$ is a set of elements of M and $\nu \in \mathbb{N}^{(\mathcal{I})}$ then we let

$$\gamma^\nu(x) = \mathop{\times}_{\alpha\in\mathcal{I}} \gamma^{\nu_\alpha}(x_\alpha)$$

which is an element of $\Gamma^d_A(M)$ with $d=|\nu|=\sum_{\alpha\in\mathcal{I}}\nu_\alpha$.

(1.1.3) *Functoriality* — $\Gamma_A(\cdot)$ is a covariant functor from the category of A-modules to the category of graded A-algebras [Rob63, Ch. III §4, p. 251].

(1.1.4) *Base change* — If A' is an A-algebra then there is a natural isomorphism $\Gamma_A(M)\otimes_A A'\to\Gamma_{A'}(M\otimes_A A')$ mapping $\gamma^d(x)\otimes_A 1$ to $\gamma^d(x\otimes_A 1)$ [Rob63, Thm. III.3, p. 262].

(1.1.5) *Multiplicative structure* — When B is an A-algebra then the multiplication of B induces a multiplication on $\Gamma^d_A(B)$ which we will denote by juxtaposition [Rob80]. This multiplication is such that $\gamma^d(x)\gamma^d(y)=\gamma^d(xy)$.

(1.1.6) *Universal property* — If M is an A-module, then the A-module $\Gamma^d_A(M)$ represents *polynomial laws* which are homogeneous of degree d [Rob63, Thm. IV.1, p. 266]. If B is an A-algebra, then the A-algebra $\Gamma^d_A(B)$ represents *multiplicative polynomial laws* which are homogeneous of degree d [Fer98, Prop. 2.5.1].

(1.1.7) *Basis* — If $(x_\alpha)_{\alpha\in\mathcal{I}}$ is a set of generators of M, then $\bigl(\gamma^\nu(x)\bigr)_{\nu\in\mathbb{N}^{(\mathcal{I})}}$ is a set of generators of $\Gamma_A(M)$. If $(x_\alpha)_{\alpha\in\mathcal{I}}$ is a basis of M then $\bigl(\gamma^\nu(x)\bigr)_{\nu\in\mathbb{N}^{(\mathcal{I})}}$ is a basis of $\Gamma_A(M)$ [Rob63, Thm. IV.2, p. 272].

(1.1.8) *Presentation* — Let $M=G/R$ be a presentation of the A-module M. Then $\Gamma_A(M)=\Gamma_A(G)/I$ where I is the ideal of $\Gamma_A(G)$ generated by the images in $\Gamma_A(G)$ of $\gamma^d(x)$ for every $x\in R$ and $d\geq 1$ [Rob63, Prop. IV.8, p. 284].

(1.1.9) Γ *and* TS — The homogeneous polynomial law $M\to\mathrm{TS}^d_A(M)$ of degree d given by $x\mapsto x^{\otimes_A d}=x\otimes_A\cdots\otimes_A x$ corresponds by the universal property (1.1.6) to an A-module homomorphism $\varphi:\Gamma^d_A(M)\to\mathrm{TS}^d_A(M)$ [Rob63, Prop. III.1, p. 254].

When M is a free A-module then φ is an isomorphism [Rob63, Prop. IV.5, p. 272]. The functors Γ^d_A and TS^d_A commute with filtered direct limits [**I**, 1.1.4, 1.2.11]. Since any flat A-module is the filtered direct limit of free A-modules [Laz69, Thm. 1.2], it thus follows that $\varphi:\Gamma^d_A(M)\to\mathrm{TS}^d_A(M)$ is an isomorphism for any flat A-module M.

Moreover by [Rob63, Prop. III.3, p. 256], there is a diagram of A-modules

$$\begin{tikzcd}
\mathrm{TS}^d_A(M) \arrow[r, hook] & \mathrm{T}^d_A(M) \arrow[d, two heads] \\
\Gamma^d_A(M) \arrow[u] & \mathrm{S}^d_A(M) \arrow[l]
\end{tikzcd}$$

such that going around the square is multiplication by $d!$. Thus if $d!$ is invertible then $\Gamma^d_A(M)\to\mathrm{TS}^d_A(M)$ is an isomorphism. In particular, this is the case when A is purely of characteristic zero, i.e., contains the field of rationals.

Let B be an A-algebra. As the law $B \to \mathrm{TS}^d_A(B)$ given by $x \mapsto x^{\otimes_A d}$ is multiplicative, it follows that the homomorphism $\varphi : \Gamma^d_A(B) \to \mathrm{TS}^d_A(B)$ is an A-algebra homomorphism. In Section 4.1 we study the properties of φ more closely.

1.2. **When is $\Gamma^d_A(M)$ generated by $\gamma^d(M)$?** $\Gamma^d_A(M)$ is not always generated by $\gamma^d(M)$ but a result due to Ferrand [Fer98, Lem. 2.3.1], cf. Proposition (1.2.4), shows that there is a finite free base change $A \hookrightarrow A'$ such that $\Gamma^d_{A'}(M \otimes_A A')$ is generated by $\gamma^d(M \otimes_A A')$. We will prove a slightly stronger statement in Proposition (1.2.2).

We let $\big(\gamma^d(M)\big)$ denote the A-submodule of $\Gamma^d_A(M)$ generated by the subset $\gamma^d(M)$.

Lemma (1.2.1). *Let A be a ring and M an A-module. There is a commutative diagram*

$$\begin{tikzcd}
\big(\gamma^d_A(M)\big) \otimes_A A' \arrow[r] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\circ"] & \Gamma^d_A(M) \otimes_A A' \arrow[d, "\psi"', "\cong"] \\
\big(\gamma^d_{A'}(M \otimes_A A')\big) \arrow[r, phantom, "\subseteq"] & \Gamma^d_{A'}(M \otimes_A A')
\end{tikzcd}$$

where ψ is the canonical isomorphism of (1.1.4). *If $A \to A'$ is a surjection or a localization then φ is surjective. In particular, if in addition $\big(\gamma^d_{A'}(M \otimes_A A')\big) = \Gamma^d_{A'}(M \otimes_A A')$ then $\big(\gamma^d_A(M)\big) \otimes_A A' \to \Gamma^d_A(M) \otimes_A A'$ is surjective.*

Proof. The morphism φ is well-defined as $\psi\big(\gamma^d(x) \otimes_A a'\big) = a'\gamma^d(x \otimes_A 1)$ if $x \in M$ and $a' \in A'$. If $A' = A/I$ then φ is clearly surjective. If $A' = S^{-1}A$ is a localization then φ is surjective since any element of $M \otimes_A A'$ can be written as $x \otimes_A (1/f)$ and $\varphi\big(\gamma^d(x) \otimes_A 1/f^d\big) = \gamma^d\big(x \otimes_A (1/f)\big)$. $\square$

Proposition (1.2.2). *Let M be an A-module. The A-module $\Gamma^d_A(M)$ is generated by the subset $\gamma^d(M)$ if the following condition is satisfied*

- (*) *For every $\mathfrak{p} \in \mathrm{Spec}(A)$ the residue field $k(\mathfrak{p})$ has at least d elements or $M_\mathfrak{p}$ is generated by one element.*

If M is of finite type, then this condition is also necessary.

Proof. Lemma (1.2.1) gives that $\big(\gamma^d_A(M)\big) = \Gamma^d_A(M)$ if and only if $\big(\gamma^d_{A_\mathfrak{p}}(M_\mathfrak{p})\big) = \Gamma^d_{A_\mathfrak{p}}(M_\mathfrak{p})$ for every $\mathfrak{p} \in \mathrm{Spec}(A)$. We can thus assume that A is a local ring and only need to consider the condition (*) for the maximal ideal $\mathfrak{m}$. If M is generated by one element then it is obvious that $\big(\gamma^d_A(M)\big) = \Gamma^d_A(M)$.

Further, any element in $\Gamma^d_A(M)$ is the image of an element in $\Gamma^d_A(M')$ for some submodule $M' \subseteq M$ of finite type. It is thus sufficient, but not necessary, that $\Gamma^d_A(M')$ is generated by $\gamma^d(M')$ for every submodule $M' \subseteq M$ of finite type. We can thus assume that M is of finite type. Lemma (1.2.1) applied with $A \twoheadrightarrow A/\mathfrak{m} = k(\mathfrak{m})$ together with Nakayama's lemma then shows that $\big(\gamma^d_A(M)\big) = \Gamma^d_A(M)$ if and only if $\big(\gamma^d_{A/\mathfrak{m}}(M/\mathfrak{m}M)\big) = \Gamma^d_{A/\mathfrak{m}}(M/\mathfrak{m}M)$. We can thus assume that $A = k$ is a field.

We will prove by induction on e that $\Gamma^e_k(M)$ is generated by $\gamma^e(M)$ when $0 \leq e \leq d$ if and only if either $\mathrm{rk}\, M \leq 1$ or $|k| \geq e$. Every element in $\Gamma^e_k(M)$ is a linear combination of elements of the form

$$\gamma^\nu(x) = \gamma^{\nu_1}(x_1) \times \gamma^{\nu_2}(x_2) \times \cdots \times \gamma^{\nu_m}(x_m).$$

where $x_i \in M$ and $|\nu| = e$. By induction $\gamma^{\nu_2}(x_2)\times\cdots\times\gamma^{\nu_m}(x_m) \in \big(\gamma^{e-\nu_1}(M)\big)$ and we can thus assume that $m = 2$ and it is enough to show that $\gamma^i(x) \times \gamma^{e-i}(y) \in \big(\gamma^e(M)\big)$ for every $x, y \in M$ and $0 \leq i \leq e$ if and only if either $\operatorname{rk} M \leq 1$ or $|k| \geq e$. If x and y are linearly dependent this is obvious. Thus we need to show that for x and y linearly independent, we have that $\gamma^i(x)\times\gamma^{e-i}(y) \in \big(\gamma^e(kx\oplus ky)\big)$ if and only if $|k| \geq e$. A basis for $\Gamma^e_k(kx \oplus ky)$ is given by $z_0, z_1, \ldots, z_e$ where $z_i = \gamma^i(x) \times \gamma^{e-i}(y)$, see (1.1.7). For any $a, b \in k$ we let

$$\xi_{a,b} := \gamma^e(ax + by) = \sum_{i=0}^{e} \gamma^i(ax) \times \gamma^{e-i}(by) = \sum_{i=0}^{e} a^i b^{e-i} z_i.$$

Then $\big(\gamma^e_k(kx \oplus ky)\big) = \Gamma^e_k(kx \oplus ky)$ if and only if $\sum_{(a,b)\in k^2} k\xi_{a,b} = \bigoplus_{i=0}^e kz_i$. Since $\xi_{\lambda a,\lambda b} = \lambda^e \xi_{a,b}$ this is equivalent to $\sum_{(a:b)\in\mathbb{P}^1_k} k\xi_{a,b} = \bigoplus_{i=0}^e kz_i$. It is thus necessary that $\big|\mathbb{P}^1_k\big| = |k| + 1 \geq e + 1$. On the other hand if $a_1, a_2, \ldots, a_e \in k^*$ are distinct then $\xi_{a_1,1}, \xi_{a_2,1}, \ldots, \xi_{a_e,1}, \xi_{1,0}$ are linearly independent. In fact, this amounts to $(1, a_i, a_i^2, a_i^3, \ldots, a_i^e)_{i=1,2,\ldots,e}$ and $(0, 0, \ldots, 0, 1)$ being linearly independent in k^{e+1}. If they are dependent then there exist a non-zero $(c_0, c_1, \ldots, c_{e-1}) \in k^e$ such that $c_0 + c_1 a_i + c_2 a_i^2 + \cdots + c_{e-1} a_i^{e-1} = 0$ for every $1 \leq i \leq e$ but this is impossible since $c_0 + c_1 x + \cdots + c_{e-1} x^{e-1} = 0$ has at most $e - 1$ solutions. □

Lemma (1.2.3). *Let $\Lambda_d = \mathbb{Z}[T]/P_d(T)$ where $P_d(T)$ is the unitary polynomial $\prod_{0\leq i<j\leq d}(T^i - T^j) - 1$. Then every residue field of Λ_d has at least $d + 1$ elements. In particular, if A is any algebra, then $A \hookrightarrow A' = A \otimes_{\mathbb{Z}} \Lambda_d$ is a faithfully flat finite extension such that every residue field of A' has at least $d + 1$ elements.*

Proof. The Vandermonde matrix $(T^{ij})_{0\leq i,j\leq d}$ is invertible in $\operatorname{End}_{\Lambda_d}(\Lambda_d^{d+1})$ since it has determinant one. Let k be a field and $\varphi : \Lambda_d \to k$ be any homomorphism. If $t = \varphi(T)$ then $(t^{ij})_{0\leq i,j\leq d}$ is invertible in $\operatorname{End}_k(k^{d+1})$ and it follows that $1, t, t^2, \ldots, t^d$ are all distinct and hence that k has at least $d + 1$ elements. □

Proposition (1.2.4). [Fer98, Lem. 2.3.1] *Let Λ_d be as in Lemma* (1.2.3). *If A is a Λ_d-algebra then $\Gamma^d_A(M)$ is generated by $\gamma^d(M)$. In particular, for every A there is a finite faithfully flat extension $A \to A'$, independent of M, such that $\Gamma^d_{A'}(M')$ is generated by $\gamma^d(M')$.*

Proof. Follows immediately from Proposition (1.2.2) and Lemma (1.2.3). □

1.3. **Generators of the ring of divided powers.** In this section we will recall some results on the degree of the generators of $\Gamma^d_A(B)$. For our purposes the results of Fleischmann [Fle98] is sufficient and we will not use the more precise and stronger statements of [Ryd07a] even though some bounds would be slightly improved then.

Definition (1.3.1) (Multidegree)**.** Let $B = A[x_1, x_2, \dots, x_r]$. We define the *multidegree* of a monomial $x^\alpha \in B$ to be α. This makes B into a $\mathbb{N}^r$-graded ring

$$B = \bigoplus_{\alpha \in \mathbb{N}^r} B_\alpha = \bigoplus_{\alpha \in \mathbb{N}^r} A x^\alpha$$

Let $\mathcal{M}$ be the A-module basis of B consisting of the monomials. Recall from paragraph (1.1.7) that a basis of $\Gamma_A(B)$ is given by the elements $\gamma^\nu(x) = \times_\alpha \gamma^{\nu_\alpha}(x^\alpha)$ for $\nu \in \mathbb{N}^{(\mathcal{M})}$. We let $\operatorname{mdeg}\bigl(\gamma^k(x^\alpha)\bigr) = k\alpha$ and $\operatorname{mdeg}(f \times g) = \operatorname{mdeg}(f) + \operatorname{mdeg}(g)$ for $f, g \in \Gamma_A(B)$. Then

$$\operatorname{mdeg}\bigl(\mathop{\times}_{\alpha} \gamma^{\nu_\alpha}(x^\alpha)\bigr) = \sum_{x^\alpha \in \mathcal{M}} \nu_\alpha \operatorname{mdeg}(x^\alpha) = \sum_{\alpha \in \mathbb{N}^r} \nu_\alpha \alpha.$$

We let $\Gamma^d_A(B)_\alpha$ be the A-module generated by basis elements $\gamma^\nu(x)$ of multidegree α. This makes $\Gamma^d_A(B) = \bigoplus_{\alpha \in \mathbb{N}^r} \Gamma^d_A(B)_\alpha$ into a $\mathbb{N}^r$-graded ring.

Definition (1.3.2) (Degree)**.** Let $B = A[x_1, x_2, \dots, x_r] = \bigoplus_{k \geq 0} B_k$ with the usual grading, i.e., B_k are the homogeneous polynomials of degree k. The graded A-algebra $C = \bigoplus_{k \geq 0} \Gamma^d_A(B_k)$ is a subalgebra of $\Gamma^d_A(B)$. If an element $f \in \Gamma^d_A(B)$ belongs to $C_k = \Gamma^d_A(B_k)$ we say that f is *homogeneous of degree k*. The *degree* of an arbitrary element $f \in \Gamma^d_A(B)$ is the smallest natural number n such that $f \in \Gamma^d_A\left(\bigoplus_{k=0}^n B_k\right)$.

Remark **(1.3.3).** Let $B = A[x_0, x_1, \dots, x_r]$ and let $C = \bigoplus_{k \geq 0} \Gamma^d_A(B_k)$ be the graded subring of $\Gamma^d_A(B)$. The degree in the previous definition is such that there is a relation between the degree of elements in C and the degree of an element in the graded localization $C_{(\gamma^d(s))}$ for $s \in B_1$. To see this, note that

$$C_{(\gamma^d(s))} = \Gamma^d_A\left(B_{(s)}\right) = \Gamma^d_A\bigl(A[x_0/s, \dots, x_r/s]\bigr).$$

We let $A[x_0/s, \dots, x_r/s]$ be graded such that x_i/s has degree 1. An element $f \in \Gamma^d_A(A[x_0/s, \dots, x_r/s])$ of degree n can then be written as $g/\gamma^d(s)^n$ where $g \in \Gamma^d_A(B_n)$ is homogeneous of degree n.

Theorem (1.3.4) ([Ric96, Prop. 2], [Ryd07a, Cor. 8.4])**.** *If $d!$ is invertible in A then $\Gamma^d_A(A[x_1, \dots, x_r])$ is generated by the elementary multisymmetric functions $\gamma^{d_1}(x_1) \times \gamma^{d_2}(x_2) \times \dots \times \gamma^{d_r}(x_r) \times \gamma^{d-d_1-d_2-\dots-d_r}(1)$, $d_i \in \mathbb{N}$ and $d_1 + d_2 + \dots + d_r \leq d$.*

Theorem (1.3.5) ([Fle98, Thm. 4.6, 4.7], [Ryd07a, Cor. 8.6])**.** *For an arbitrary ring A, the A-algebra $\Gamma^d_A(A[x_1, \dots, x_r])$ is generated by $\gamma^d(x_1), \gamma^d(x_2), \dots, \gamma^d(x_r)$ and the elements $\gamma^k(x^\alpha) \times \gamma^{d-k}(1)$ with $k\alpha \leq (d-1, d-1, \dots, d-1)$. Further, there is no smaller multidegree bound and if $d = p^s$ for some prime p not invertible in A, then $\Gamma^d_A(A[x_1, \dots, x_r])$ is not generated by elements of strictly smaller multidegree.*

Theorems (1.3.4) and (1.3.5) give the following degree bound:

Corollary (1.3.6). *Let A be a ring and $B = k[x_1, x_2, \dots, x_r]$. Then $\Gamma^d_A(B)$ is generated by elements of degree at most $\max\bigl(1, r(d-1)\bigr)$. If $d!$ is invertible in A, then $\Gamma^d_A(B)$ is generated by elements of degree one.*

2. Weighted projective schemes and quotients by finite groups

In this section we review the definition and the basic results on weighted projective schemes. We will in particular focus on weighted projective structures which are *covered in degree one*. By this, we mean that the sections of $\mathcal{O}(1)$ give an affine cover of the weighted projective scheme. We then recall the construction, using invariant theory, of a geometric quotient of a projective scheme by a finite group. Finally, we discuss the failure of a geometric quotient to commute with arbitrary base change and closed immersions.

2.1. **Remarks on projectivity.** We will follow the definitions in EGA. In particular, very ample, ample, quasi-projective and projective will have the meanings of [$\mathrm{EGA}_{\mathrm{II}}$, §4.4, §4.6, §5.3, §5.5]. By definition, a morphism $q : X \to S$ is quasi-projective if it is of finite type and there exists an invertible $\mathcal{O}_X$-sheaf $\mathcal{L}$ ample with respect to q. Note that this does *not* imply that X is a subscheme of $\mathbb{P}_S(\mathcal{E})$ for some quasi-coherent $\mathcal{O}_S$-module $\mathcal{E}$. However, if S is quasi-compact and quasi-separated then there is a quasi-coherent $\mathcal{O}_S$-module of finite type $\mathcal{E}$ and an immersion $X \hookrightarrow \mathbb{P}(\mathcal{E})$ [$\mathrm{EGA}_{\mathrm{II}}$, Prop. 5.3.2]. Similarly, a projective morphism is always quasi-projective and proper but the converse only holds if S is quasi-compact and quasi-separated.

Furthermore, if $q : X \to S$ is a projective morphism and $\mathcal{L}$ a very ample invertible sheaf on X then $\mathcal{L}$ does not necessarily correspond to a closed embedding into a projective space over S. We always have a closed embedding $X \hookrightarrow \mathbb{P}(q_*\mathcal{L})$ as q is proper [$\mathrm{EGA}_{\mathrm{II}}$, Prop. 4.4.4] but $q_*\mathcal{L}$ need not be of finite type. If S is *locally noetherian* however, then $q_*\mathcal{L}$ is of finite type [$\mathrm{EGA}_{\mathrm{III}}$, Thm. 3.2.1]. If S is quasi-compact and quasi-separated then we can find a sub-$\mathcal{O}_S$-module of finite type $\mathcal{E}$ of $q_*\mathcal{L}$ such that we have a closed immersion $i : X \hookrightarrow \mathbb{P}(\mathcal{E})$ and such that $\mathcal{L} = i^*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$.

We will also need the following stronger notion of projectivity introduced by Altman and Kleiman in [AK80, §2]. Our definition differs slightly from theirs as we do not require strongly projective morphisms to be of finite *presentation.*

Definition (2.1.1). A morphism $X \to S$ is *strongly projective* (resp. *strongly quasi-projective*) if it is of finite type and factors through a closed immersion (resp. an immersion) $X \hookrightarrow \mathbb{P}_S(\mathcal{L})$ where $\mathcal{L}$ is a locally free $\mathcal{O}_S$-module of constant rank.

Remark **(2.1.2).** A strongly (quasi-)projective morphism is (quasi-)projective and the converse holds when S is quasi-compact, quasi-separated and admits an ample sheaf, e.g., S affine [AK80, Ex. 2.2 (i)]. In fact, in this case there is an embedding $X \hookrightarrow \mathbb{P}^n_S$ and thus the notions of projective and strongly projective also agree with the definitions in [Har77].

2.2. **Weighted projective schemes.**

Definition (2.2.1). Let S be a scheme. A *weighted projective scheme* over S is an S-scheme X together with a quasi-coherent graded $\mathcal{O}_S$-algebra $\mathcal{A}$ of *finite type*, not necessarily generated by degree one elements, such that $X = \operatorname{Proj}_S(\mathcal{A})$. We let as usual $\mathcal{O}_X(n) = \widetilde{\mathcal{A}(n)}$ for any $n \in \mathbb{Z}$.

If $\mathcal{A}$ is generated by degree one elements then the sheaves $\mathcal{O}_X(n)$ are invertible for any integer n and very ample if n is positive. Furthermore, there is then a canonical isomorphism $\mathcal{O}_X(m) \otimes_{\mathcal{O}_X} \mathcal{O}_X(n) = \mathcal{O}_X(m+n)$. All these properties may fail if $\mathcal{A}$ is not generated by degree one elements.

It can however be shown, cf. Corollary (2.2.4), that if S is quasi-compact then $q : X \to S$ is projective. To be precise, there is a positive integer n such that $\mathcal{O}_X(n)$ is invertible, the homomorphism $q^*\mathcal{A}_n \to \mathcal{O}_X(n)$ is surjective and $i_n : X \to \mathbb{P}(\mathcal{A}_n)$ is a closed immersion. In particular, $\mathcal{O}_X(n) = i_n^*\mathcal{O}_{\mathbb{P}(\mathcal{A}_n)}(1)$ is very ample. Another consequence is that if X is a weighted projective scheme over an arbitrary scheme S then $X \to S$ is proper.

We will give a demonstration of the projectivity of $X \to S$ when S is quasi-compact and also show some properties of the sheaves $\mathcal{O}_X(n)$. The results are somewhat weaker than those in [BR86, §4] but we also give stronger results when X is covered in degree one.

The following lemma is an explicit form of [EGA$_{\mathrm{II}}$, Lem. 2.1.6].

Lemma (2.2.2). *If B is a graded A-algebra generated by elements $f_1, f_2, \dots, f_s \in B$ of degrees $d_1, d_2, \dots, d_s$ and l is the least common multiple of $d_1, d_2, \dots, d_s$ then*

(i) *$B_{n+l} = (B_n B_l)$ for every $n \geq (s-1)(l-1)$.*
(ii) *$B_{kn} = (B_n)^k$ for every $k \geq 0$ if $n = al$ with $a \geq s-1$.*

Proof. Clearly B_k is generated by $f_1^{a_1} f_2^{a_2} \dots f_s^{a_s}$ such that $\sum_i a_i d_i = k$. Let $g_i = f_i^{l/d_i} \in B_l$. If $k \geq s(l-1)+1$ and $f = f_1^{a_1} f_2^{a_2} \cdots f_s^{a_s} \in B_k$ then $g_i | f$ for some i which shows (i). (ii) follows easily from (i). □

Proposition (2.2.3) (cf. [BR86, Cor. 4A.5, Thm. 4B.7])**.** *Let A be a ring and let B be a graded A-algebra generated by a finite number of elements $f_1, f_2, \dots, f_s$ of degrees $d_1, d_2, \dots, d_s$. Let l be the least common multiple of the d_i's. Let $S = \operatorname{Spec}(A)$, $X = \operatorname{Proj}(B)$ and $\mathcal{O}_X(n) = \widetilde{B(n)}$. Then*

(i) *$X = \bigcup_{f \in B_n} \mathrm{D}_+(f)$ if $n = al$ and $a \geq 1$.*
(ii) *$\mathcal{O}_X(n)$ is invertible if $n = al$ and $a \in \mathbb{Z}$.*
(iii) *$\mathcal{O}_X(n)$ is ample and generated by global sections if $n = al$ and $a \geq 1$.*
(iv) *The canonical homomorphism $\mathcal{O}_X(al) \otimes \mathcal{O}_X(n) \to \mathcal{O}_X(al+n)$ is an isomorphism for every $a, n \in \mathbb{Z}$.*
(v) *If $n = al$ with $a \geq 1$ then there is a canonical morphism $i_n : X \to \mathbb{P}(B_n)$. If $a \geq \max\{1, s-1\}$ then i_n is a closed immersion and $\mathcal{O}_X(n) = i_n^*\mathcal{O}_{\mathbb{P}(B_n)}(1)$ is very ample relative to S.*
(vi) *$\mathcal{O}_X(n)$ is generated by global sections if $n \geq (s-1)(l-1)$.*

Proof. (i) is trivial as $X = \bigcup_{i=1}^{s} \mathrm{D}_+(f_i) = \bigcup_{i=1}^{s} \mathrm{D}_+\big(f_i^{al/d_i}\big)$ if $a \geq 1$, cf [EGA$_{\text{II}}$, Cor. 2.3.14]. Note that if $f \in B_l$ then

$$B_f = \big(B_{(f)} \oplus B(1)_{(f)} \oplus \cdots \oplus B(l-1)_{(f)}\big)[f, f^{-1}]. \tag{2.2.3.1}$$

Thus $\Gamma\big(\mathrm{D}_+(f), \mathcal{O}_X(al)\big) = B(al)_{(f)} = B_{(f)} f^a$ is a free $B_{(f)}$-module of rank one which shows (ii).

(iii) If $a \geq 1$ then $\big(\mathrm{D}_+(f)\big)_{f \in B_{al}}$ is an affine cover of X. As $\mathcal{O}_X(al)$ is an invertible sheaf it is thus generated by global sections and ample by definition, cf. [EGA$_{\text{II}}$, Def. 4.5.3 and Thm. 4.5.2 a$'$)].

(iv) It is enough to show that the homomorphism $\mathcal{O}_X(al) \otimes \mathcal{O}_X(n) \to \mathcal{O}_X(al+n)$ is an isomorphism locally over $\mathrm{D}_+(f)$ with $f \in B_l$. Locally this homomorphism is $B(al)_{(f)} \otimes_{B_{(f)}} B(n)_{(f)} \to B(al+n)_{(f)}$ which is an isomorphism by equation (2.2.3.1)

(v) If $n = al$ with $a \geq 1$ then by (i) the morphism $i_n : X \to \mathbb{P}(B_n)$ is everywhere defined. If in addition $a \geq s-1$ then $B^{(n)}$ is generated by degree one elements by Lemma (2.2.2) (ii). Thus we have a closed immersion $X = \mathrm{Proj}(B) \cong \mathrm{Proj}(B^{(n)}) \hookrightarrow \mathbb{P}(B_n)$.

(vi) Assume that $n \geq (s-1)(l-1)$, then $B_{n+kl} = (B_n B_l^k)$ for any positive integer k by Lemma (2.2.2) (i). If $f \in B_l$ and $b \in B(n)_{(f)}$, then $b = b'/f^k$ for some $b' \in B_{n+kl} = \big(B_n B_l^k\big)$ and thus $b \in \big(B_{(f)} B_n\big)$. This shows that $\mathcal{O}_X(n)$ is generated by global sections as $B_n \subseteq \Gamma\big(\mathrm{D}_+(f), \mathcal{O}_X(n)\big)$. ☐

Corollary (2.2.4) ([EGA$_{\text{II}}$, Cor. 3.1.11])**.** *If S is quasi-compact and $X = \mathrm{Proj}_S(\mathcal{A})$ is a weighted projective scheme then there exists a positive integer n such that $X \to \mathbb{P}(\mathcal{A}_n)$ is everywhere defined and a closed immersion. In particular X is projective and $\mathcal{O}_X(n)$ is very ample relative to S.*

Proof. Let $\{S_i\}$ be a finite affine cover of S and let $A_i = \Gamma(S_i, \mathcal{O}_S)$ and $B_i = \Gamma(S_i, \mathcal{A})$. Then as B_i is a finitely generated graded A_i-algebra, there is by Proposition (2.2.3) a positive integer n_i such that $X \times_S S_i \to \mathbb{P}\big((B_i)_{n_i}\big)$ is defined and a closed immersion. Choosing n as the least common multiple of the n_i's we obtain a closed immersion $X \hookrightarrow \mathbb{P}(\mathcal{A}_n)$. ☐

Remark **(2.2.5).** Note that (2.2.3) (iv), (v), (vi) implies that the following are equivalent:

(i) $\mathcal{O}_X(n)$ is invertible for all $0 < n < l$.
(ii) $\mathcal{O}_X(n)$ is invertible for all n.
(iii) $\mathcal{O}_X(n)$ is very ample for all sufficiently large n.

As (i) is easily seen to not hold in many examples in particular (iii) is not always true.

The following condition will be important later on as it is satisfied for $\mathrm{Sym}^d(X/S)$ for X/S quasi-projective. Note that in the remainder of this section we do not assume that $\mathcal{A}$ is finitely generated. In particular, $\mathrm{Proj}_S(\mathcal{A})$ need not be a weighted projective space.

Definition (2.2.6). Let S be a scheme, $\mathcal{A}$ a graded quasi-coherent $\mathcal{O}_S$-algebra and $X = \mathrm{Proj}_S(\mathcal{A})$. If there is an affine cover (S_α) of S such that $X \times_S S_\alpha$ is covered by $\bigcup_{f \in \Gamma(S_\alpha, \mathcal{A}_1)} \mathrm{D}_+(f)$, then we say that X/S is *covered in degree one.*

Proposition (2.2.7). *Let A be a ring and let B be a graded A-algebra generated by elements of degree $\leq d$. Let $S = \mathrm{Spec}(A)$, $X = \mathrm{Proj}(B)$ and $\mathcal{O}_X(n) = \widetilde{B(n)}$. If X/S is covered in degree one then*

(i) $X = \bigcup_{f \in B_n} \mathrm{D}_+(f)$ *if* $n \geq 1$.
(ii) $\mathcal{O}_X(n)$ *is invertible for* $n \in \mathbb{Z}$ *and ample and generated by global sections if* $n \geq 1$.
(iii) $\mathcal{O}_X(m) \otimes \mathcal{O}_X(n) \cong \mathcal{O}_X(m+n)$ *for every* $m, n \in \mathbb{Z}$.
(iv) *The canonical morphism* $i_n : X \to \mathbb{P}(B_n)$ *is defined for every* $n \geq 1$. *If* $n \geq d$ *then* i_n *is a closed immersion and* $\mathcal{O}_X(n) = i_n^* \mathcal{O}_{\mathbb{P}(B_n)}(1)$ *is very ample relative* S.

Proof. (i) is equivalent to X/S being covered in degree one. Using the cover $X = \bigcup_{f \in B_1} \mathrm{D}_+(f)$ instead of the cover $X = \bigcup_{f \in B_l} \mathrm{D}_+(f)$ we may then prove (ii) and (iii) exactly as (ii), (iii) and (iv) in Proposition (2.2.3).

(iv) Let $n \geq d$ and let B' be the sub-A-algebra of B generated by B_n. It is enough to show that the inclusion $B' \hookrightarrow B$ induces an isomorphism $\mathrm{Proj}(B) \cong \mathrm{Proj}(B')$. We will show this using the cover $X = \bigcup_{f \in B_1} \mathrm{D}_+(f^n)$. Let $f \in B_1$ and $g \in B_{(f^n)}$ such that $g = b/f^{nk}$ for some $b \in B_{nk}$. To show that $g \in B'_{(f^n)}$ we can assume that $b = b_1 b_2 \dots b_s$ is a product of elements of degrees $d_i \leq d$, as every element of B_{nk} are sums of such. Then $g = \left(\prod_{i=1}^s b_i f^{n-d_i}\right) / f^{ns}$ which is an element of $B'_{(f^n)}$. □

Corollary (2.2.8). *Let S be any scheme and $\mathcal{A}$ a graded quasi-coherent $\mathcal{O}_S$-algebra such that $\mathcal{A}$ is generated by elements of degree at most d. Let $X = \mathrm{Proj}(\mathcal{A})$, $\mathcal{O}_X(n) = \widetilde{\mathcal{A}(n)}$ and assume that X/S is covered in degree one. Then*

(i) $\mathcal{O}_X(n) = \mathcal{O}_X(1)^{\otimes n}$ *is invertible for every* $n \in \mathbb{Z}$.
(ii) *If* $n \geq 1$ *then* $\mathcal{O}_X(n)$ *is ample and* $q^* \mathcal{A}_n \to \mathcal{O}_X(n)$ *is surjective.*
(iii) *For every* $n \geq 1$ *the canonical morphism* $i_n : X \to \mathbb{P}(\mathcal{A}_n)$ *is everywhere defined. If* $n \geq d$ *it is a closed immersion.*

In particular, if $X = \mathrm{Proj}_S(\mathcal{A})$ also is a weighted projective scheme, i.e., if $\mathcal{A}$ is of finite type, then X is projective.

Example (2.2.9) (Standard weighted projective spaces)**.** Let $A = k$ be an algebraically closed field of characteristic zero and $B = k[x_0, x_1, \dots, x_r]$. Choose positive integers $d_0, d_1, \dots, d_r$ and consider the action of $G = \mu_{d_0} \times \dots \times \mu_{d_r} \cong \mathbb{Z}/d_0\mathbb{Z} \times \dots \times \mathbb{Z}/d_r\mathbb{Z}$ on B given by $(n_0, n_1, \dots, n_r) \cdot x_i = \xi_{d_i}^{n_i} x_i$ where ξ_{d_i} is a ${d_i}^{\text{th}}$ primitive root of unity. Then $B^G = k[x_0^{d_0}, x_1^{d_1}, \dots, x_r^{d_r}]$ and $\mathrm{Proj}\left(B^G\right)$ is a *weighted projective space* of type $(d_0, d_1, \dots, d_r)$.

It can be seen, cf. Proposition (2.3.4), that $\mathrm{Proj}\left(B^G\right)$ is the geometric quotient of $\mathrm{Proj}(B) = \mathbb{P}^r$ by G. More generally, if S is a noetherian scheme and X/S is projective with an action of a finite group G linear with respect to a very ample sheaf $\mathcal{O}_X(1)$, then a geometric quotient X/G exists and can be given a structure as a weighted projective scheme.

The weighted projective space $\operatorname{Proj}\left(B^G\right)$ is often denoted $\mathbb{P}(d_0, d_1, \dots, d_r)$. It can also be constructed as the quotient of $\mathbb{A}^{r+1} - 0$ by $\mathbb{G}_m$ where $\mathbb{G}_m$ acts on $\mathbb{A}^{r+1}$ by $\lambda \cdot x_i = \lambda^{d_i} x_i$. The closed points of $\mathbb{P}(d_0, d_1, \dots, d_r)$ are thus $\{x = (x_0 : x_1 : \dots : x_r)\} = k^{r+1}/\sim$ where $x \sim y$ if there is a $\lambda \in k^*$ such that $\lambda^{d_i} x_i = y_i$ for every i.

2.3. **Quotients of projective schemes by finite groups.** Let X be an S-scheme and G a discrete group acting on X/S, i.e., there is a group homomorphism $G \to \operatorname{Aut}_S(X)$. In the category of ringed spaces we can construct a quotient $Y = (X/G)_{\mathrm{rs}}$ as following. Let Y as a topological space be X/G with the quotient topology, and quotient map $q : X \to Y$. Further let the sheaf of sections $\mathcal{O}_Y$ be the subsheaf $(q_*\mathcal{O}_X)^G \hookrightarrow q_*\mathcal{O}_X$ of G-invariant sections. Note that G acts on $q_*\mathcal{O}_X$ since for any open subset $U \subseteq Y$ the inverse image $q^{-1}(U)$ is G-stable and hence has an induced action of G. Thus we obtain a ringed S-space $(Y, \mathcal{O}_Y)$ together with a morphism of ringed S-spaces $q : X \to Y$. The ringed space $(Y, \mathcal{O}_Y)$ is not always a scheme, in fact not always even a locally ringed space. But when it exists as a scheme it is called the *geometrical quotient* and is also the *categorial quotient* in the category of schemes over S. For general existence results we refer to [Ryd07b]. The existence of a geometric quotient of an affine schemes by a finite group is not difficult to show:

Proposition (2.3.1) ([SGA$_1$, Exp. V, Prop. 1.1, Cor. 1.5])**.** *Let S be a scheme, $\mathcal{A}$ a quasi-coherent sheaf of $\mathcal{O}_S$-algebras and $X = \operatorname{Spec}_S(\mathcal{A})$. An action of G on X/S induces an action of G on $\mathcal{A}$. If G is a finite group then $Y = \operatorname{Spec}_S\left(\mathcal{A}^G\right)$ is the geometric quotient of X by G. If S is* locally noetherian *and $X \to S$ is of finite type, then $Y \to S$ is of finite type.*

From this local result it is not difficult to show the following result:

Theorem (2.3.2) ([SGA$_1$, Exp. V, Prop. 1.8])**.** *Let $f : X \to S$ be a morphism of arbitrary schemes and G a finite discrete group acting on X by S-morphisms. Assume that every G-orbit of X is contained in an affine open subset. Then the geometrical quotient $q : X \to Y = X/G$ exists as a scheme.*

It can also be shown, from general existence results, that if X/S is separated then this is also a necessary condition [Ryd07b, Rmk. 4.9].

Remark **(2.3.3).** If $X \to S$ is quasi-projective, then every G-orbit is contained in an affine open set. In fact, we can assume that $S = \operatorname{Spec}(A)$ is affine and thus that we have an embedding $X \hookrightarrow \mathbb{P}^n_S$. For any orbit Gx we can then choose a section $f \in \mathcal{O}_{\mathbb{P}^n}(m)$ for some sufficiently large m such that $\mathrm{V}(f)$ does not intersect Gx. The affine subset $\mathrm{D}(f)$ then contains the orbit Gx. More generally [EGA$_{\mathrm{II}}$, Cor. 4.5.4] shows that every finite set, in particulary every G-orbit, is contained in an affine open set if X/S is such that there exists an ample invertible sheaf on X relative to S.

In Corollary (2.3.6) we will show that if S is a *noetherian* scheme and $X \to S$ is (quasi-)projective, then so is $X/G \to S$. In fact if X is projective we will give a weighted projective structure on X/G.

Proposition (2.3.4). *Let S be a scheme and let $\mathcal{A} = \bigoplus_{d\geq 0} \mathcal{A}_d$ be a graded quasi-coherent $\mathcal{O}_S$-algebra, generated by degree one elements. Let G be a finite group acting on $\mathcal{A}$ by graded $\mathcal{O}_S$-algebra automorphisms. Then G acts on $X = \mathrm{Proj}_S(\mathcal{A})$ linearly with respect to $\mathcal{O}_X(1)$. As X admits a very ample invertible sheaf relative to S, a geometric quotient $Y = X/G$ exists, cf. Remark* (2.3.3). *There is an isomorphism $Y \cong \mathrm{Proj}_S\left(\mathcal{A}^G\right)$ and under this isomorphism, the quotient map $q : X \to Y$ is induced by $\mathcal{A}^G \hookrightarrow \mathcal{A}$.*

Proof. Everything is local over S so we can assume that $S = \mathrm{Spec}(A)$, $\mathcal{A} = \widetilde{B}$ and $X = \mathrm{Proj}(B)$. We can cover X by G-stable affine subsets of the form $\mathrm{D}_+(f)$ with $f \in B^G$ homogeneous. In fact, if Z is a G-orbit of X then the demonstration of [EGA$_{\mathrm{II}}$, Cor. 4.5.4] shows that there is a homogeneous $f' \in B$ such that $Z \subseteq \mathrm{D}_+(f')$. If we let $f = \prod_{\sigma\in G} \sigma(f')$, then $Z \subseteq \mathrm{D}_+(f)$ and $f \in B^G$ is homogeneous. Over such an open set we have that

$$X|_{\mathrm{D}_+(f)}/G = \mathrm{Spec}\big(\big(B_{(f)}\big)^G\big) = \mathrm{Spec}\big((B^G)_{(f)}\big) = \mathrm{Proj}\left(B^G\right)|_{\mathrm{D}_+(f)}.$$

It is thus clear that $Y = \mathrm{Proj}\left(B^G\right)$. □

Remark **(2.3.5).** Note that $\mathcal{A}^G$ is not always generated by $\mathcal{A}_1^G$ even though $\mathcal{A}$ is generated by $\mathcal{A}_1$. Also, if $S = \mathrm{Spec}(A)$ is affine and $\mathcal{A} = \widetilde{B}$, we may not be able to cover $X = \mathrm{Proj}(B)$ with G-stable affine subsets of the form $\mathrm{D}_+(f)$ with $f \in B_1^G$. This is demonstrated by example (2.2.9) if we choose $d_i > 1$ for some i.

Corollary (2.3.6) ([Knu71, Ch. IV, Prop. 1.5]). *Let S be* noetherian, *$X \to S$ be projective (resp. quasi-projective) and G a finite group acting on X by S-morphisms. Then the geometrical quotient X/G is projective (resp. quasi-projective).*

Proof. Let $X \hookrightarrow (X/S)^m = X \times_S X \times_S \cdots \times_S X$ be the closed immersion given by $x \to (\sigma_1 x, \sigma_2 x, \dots, \sigma_m x)$ where $G = \{\sigma_1, \sigma_2, \dots, \sigma_m\}$. As $X \to S$ is quasi-projective and S is noetherian, there is an immersion $X \hookrightarrow \mathbb{P}_S(\mathcal{E})$ for some quasi-coherent $\mathcal{O}_S$-module of finite type $\mathcal{E}$, see [EGA$_{\mathrm{II}}$, Prop. 5.3.2]. This immersion together with the immersion $X \hookrightarrow (X/S)^m$ given above, gives a G-equivariant immersion $X \hookrightarrow \big(\mathbb{P}_S(\mathcal{E})/S\big)^n$ if we let G permute the factors of $\big(\mathbb{P}_S(\mathcal{E})/S\big)^n$. Following this immersion by the Segre embedding we get a G-equivariant immersion $f : X \hookrightarrow \mathbb{P}_S(\mathcal{E}^{\otimes m})$ where G acts linearly on $\mathbb{P}_S(\mathcal{E}^{\otimes m})$, i.e., by automorphisms of $\mathcal{E}^{\otimes m}$.

Let $Y = \overline{f(X)}$ be the schematic image of f. As Y is clearly G-stable we have an action of G on Y and a geometric quotient $q : Y \to Y/G$. Then, as $X \hookrightarrow Y$ is an open immersion and q is open, we have that $X/G = (Y/G)|_{q(Y)}$. Thus it is enough to show that Y/G is projective. Let $\mathcal{A} = \mathrm{S}(\mathcal{E}^{\otimes m})/I$ such that $Y = \mathrm{Proj}(\mathcal{A})$. Then there is an action of G on $\mathcal{A}_1$ which induces the action Y. By Proposition (2.3.4) we have that $Y/G = \operatorname{Proj}\left(\mathcal{A}^G\right)$. The scheme Y/G is a weighted projective scheme as $\mathcal{A}^G$ is an $\mathcal{O}_S$-algebra of finite type by Proposition (2.3.1). It then follows by Corollary (2.2.4) that Y/G is projective. $\square$

2.4. **Finite quotients, base change and closed subschemes.** A geometric quotient is always *uniform*, i.e., it commutes with flat base change [GIT, Rmk. (7), p. 9]. It is also a *universal topological quotient*, i.e., the fibers corresponds to the orbits and the quotient has the quotient topology and this holds after any base change. However, in positive characteristic a geometric quotient is not necessarily a *universal* geometric quotient, i.e., it need not commute with arbitrary base change. This is shown by the following example:

Example (2.4.1). Let $X = \operatorname{Spec}(B)$, $S = \operatorname{Spec}(A)$, $S' = \operatorname{Spec}(A/I)$ with $A = k[\epsilon]/\epsilon^2$ where k is a field of characteristic $p > 0$, $B = k[\epsilon, x]/(\epsilon^2, \epsilon x)$ and $I = (\epsilon)$. We have an action of $G = \mathbb{Z}/p = \langle\tau\rangle$ on B given by $\tau(x) = x + \epsilon$ and $\tau(\epsilon) = \epsilon$. Then $\tau(x^n) = x^n$ for all $n \geq 2$ and thus $B^G = k[\epsilon, x^2, x^3]/(\epsilon^2, \epsilon x^2, \epsilon x^3)$. Further, we have that $(B \otimes_A A')^G = k[x]$ and $B^G \otimes_A A' = k[x^2, x^3]$.

Recall that a morphism of schemes is a *universal homeomorphism* if the underlying morphism of topological spaces is a homeomorphism after any base change.

Proposition (2.4.2) ([EGA$_{\text{IV}}$, Cor. 18.12.11])**.** *Let $f : X \to Y$ be a morphism of schemes. Then f is a universal homeomorphism if and only if f is integral, universally injective and surjective.*

Proposition (2.4.3). *Let X/S be a scheme with an action of a finite group G such that every G-orbit of X is contained in an affine open subset. Let $S' \to S$ be any morphism and let $X' = X \times_S S'$. Then geometric quotients $q : X \to X/G$ and $r : X' \to X'/G$ exists. Let $(X/G)' = (X/G) \times_S S'$. As r is a categorical quotient we have a canonical morphism $X'/G \to (X/G)'$. This morphism is a universal homeomorphism.*

Proof. The geometric quotients q and r exists by Theorem (2.3.2). As q and r are universal topological quotients it follows that $X'/G \to (X/G)'$ is universally bijective. As $X' \to X'/G$ is surjective and $X' \to (X/G)'$ is universally open it follows that $X'/G \to (X/G)'$ is universally open and hence a universal homeomorphism. $\square$

If G acts on X and $U \subseteq X$ is a G-stable open subscheme, then U/G is an open subscheme of X/G. In fact, U/G is the image of U by the open morphism $q : X \to X/G$. If $Z \hookrightarrow X$ is a closed G-stable subscheme, then Z/G is not always the image of Z by q. In fact, Z/G need not even be a subscheme of X/G. We have the following result:

Proposition (2.4.4). *Let G be a finite group, X/S a scheme with an action of G such that the geometric quotient $q : X \to X/G$ exists. Let $Z \hookrightarrow X$ be a closed G-stable subscheme. Then the geometric quotient $r : Z \to Z/G$ exist. Let*

$q(Z)$ be the scheme-theoretic image of the morphism $Z \hookrightarrow X \to X/G$. As r is a categorical quotient, the morphism $Z \to q(Z) \hookrightarrow X/G$ factors canonically as $Z \to Z/G \to q(Z) \hookrightarrow X/G$. The morphism $Z/G \to q(Z)$ is a schematically dominant universal homeomorphism.

Proof. As Z/G and $q(Z)$ both are universal topological quotients of Z, the canonical morphism $Z/G \to q(Z)$ is universally bijective. Since $Z \to q(Z)$ is universally open and $Z \to Z/G$ is surjective we have that $Z/G \to q(Z)$ is universally open and thus a universal homeomorphism. Further as $Z \to q(Z)$ is schematically dominant the morphism $Z/G \to q(Z)$ is also schematically dominant. □

Corollary (2.4.5). *Let G and X/S be as in Proposition* (2.4.4). *There is a canonical universal homeomorphism $(X_{\mathrm{red}})/G \to (X/G)_{\mathrm{red}}$.*

We can say even more about the exact structure of $Z/G \to q(Z)$. For ease of presentation we state the result in the affine case.

Proposition (2.4.6). *Let A be a ring with an action by a finite group G and let $I \subset A$ be a G-stable ideal. Let $X = \operatorname{Spec}(A)$ and $Z = \operatorname{Spec}(A/I)$. Then $Z/G = \operatorname{Spec}\big((A/I)^G\big)$ and $q(Z) = \operatorname{Spec}\left(A^G/I^G\right)$. We have an injection $A^G/I^G \hookrightarrow (A/I)^G$. If $f \in (A/I)^G$ then there is an $n \mid \operatorname{card}(G)$ such that $f^n \in A^G/I^G$. To be more precise we have that*

- (i) *If A is a $\mathbb{Z}_{(p)}$-algebra with p a prime, e.g., a local ring with residue field k or a k-algebra with $\operatorname{char} k = p$, then n can be chosen as a power of p.*
- (ii) *If A is purely of characteristic zero, i.e., a $\mathbb{Q}$-algebra, then $A^G/I^G \hookrightarrow (A/I)^G$ is an isomorphism.*

Proof. Let $f \in A$ such that its image $\overline{f} \in A/I$ is G-invariant. To show that $\overline{f}^n \in A^G/I^G$ for some positive integer n it is enough to show that $\overline{f}^n \in A^G/I^G \otimes_{\mathbb{Z}} \mathbb{Z}_{\mathfrak{p}}$ for every $\mathfrak{p} \in \operatorname{Spec}(\mathbb{Z})$. As $\mathbb{Z} \to \mathbb{Z}_{\mathfrak{p}}$ is flat, we have that

$$\begin{aligned} A^G \otimes_{\mathbb{Z}} \mathbb{Z}_{\mathfrak{p}} &= (A \otimes_{\mathbb{Z}} \mathbb{Z}_{\mathfrak{p}})^G \\ I^G \otimes_{\mathbb{Z}} \mathbb{Z}_{\mathfrak{p}} &= (I \otimes_{\mathbb{Z}} \mathbb{Z}_{\mathfrak{p}})^G \\ A^G/I^G \otimes_{\mathbb{Z}} \mathbb{Z}_{\mathfrak{p}} &= (A \otimes_{\mathbb{Z}} \mathbb{Z}_{\mathfrak{p}})^G/(I \otimes_{\mathbb{Z}} \mathbb{Z}_{\mathfrak{p}})^G \\ (A/I)^G \otimes_{\mathbb{Z}} \mathbb{Z}_{\mathfrak{p}} &= (A/I \otimes_{\mathbb{Z}} \mathbb{Z}_{\mathfrak{p}})^G. \end{aligned}$$

Thus we can assume that A is a $\mathbb{Z}_{\mathfrak{p}}$-algebra.

Let q be the characteristic exponent of $\mathbb{Z}_{\mathfrak{p}}/\mathfrak{p}\mathbb{Z}_{\mathfrak{p}}$, i.e., $q = p$ if $\mathfrak{p} = (p)$, $p > 0$ and $q = 1$ if $\mathfrak{p} = (0)$. Choose positive integers k and m such that $\operatorname{card}(G) = q^k m$ and $q \nmid m$ if $q \neq 1$. Then choose a Sylow subgroup H of G of order q^k, or $H = (e)$ if $q = 1$, and let $\sigma_1 H, \sigma_2 H, \dots, \sigma_m H$ be its cosets. Then

$$g = \frac{1}{m} \sum_{i=1}^{m} \prod_{\sigma \in \sigma_i H} \sigma(f)$$

is G-invariant and its image $\overline{g} \in A^G/I^G$ maps to $\overline{f}^{q^k} \in (A/I)^G$. □

Proposition (2.4.6) can also be extended to the case where G is any reductive group [GIT, Lem. A.1.2].

Remark **(2.4.7).** Let X/S be a scheme with an action of a finite group G with geometric quotient $q : X \to X/G$. Then

(i) If S is a $\mathbb{Q}$-scheme and $S' \to S$ is any morphism then $(X \times_S S')/G = X/G \times_S S'$.
(ii) If S is arbitrary and $U \subseteq X$ is an open immersion then $U/G = q(U)$.
(iii) If S is a $\mathbb{Q}$-scheme and $Z \hookrightarrow X$ is a closed immersion then $Z/G = q(Z)$.
(iv) If S is a $\mathbb{Q}$-scheme then $(X/G)_{\text{red}} = X_{\text{red}}/G$.

(ii) follows from the uniformity of geometric quotients, (i) and (iv) follows from the universality of geometric quotients in characteristic zero and (iii) follows from Proposition (2.4.6).

Statement (iii) can also be proven as follows. We can assume that $X = \operatorname{Spec}(A)$ is affine. Then the homomorphism $A^G \hookrightarrow A$ has an A^G-module retraction, the Reynolds-operator R, given by $R(a) = \frac{1}{\operatorname{card}(G)} \sum_{\sigma \in G} \sigma(a)$. This implies that $A^G \hookrightarrow A$ is *universally injective*, i.e., injective after tensoring with any A-module M. In particular $A^G \hookrightarrow A$ is *cyclically pure*, i.e., $I^G A = I$, where $I^G = I \cap A^G$, for any ideal $I \subseteq A$. If we let $S = \operatorname{Spec}\left(A^G\right)$ and $S' = \operatorname{Spec}\left(A^G/I^G\right)$ then $Z = X \times_S S' = \operatorname{Spec}(A/I)$ and (iii) follows from (i).

3. The parameter spaces

In this section we define the symmetric product $\operatorname{Sym}^d(X/S)$, the scheme of divided powers $\Gamma^d(X/S)$, and the Chow scheme of zero-cycles $\operatorname{Chow}_{0,d}(X \hookrightarrow \mathbb{P}(\mathcal{E}))$. We show that when X/S is a projective bundle, then $\operatorname{Sym}^d(X/S)$ is projective and we essentially obtain a bound on the generators of the natural weighted projective structure, cf. Theorem (3.1.12). As a corollary, we show that if X/S is projective then so is $\Gamma^d(X/S)$, cf. Theorem (3.2.10). If $\mathcal{A}$ is a graded $\mathcal{O}_S$-algebra and $X = \operatorname{Proj}(\mathcal{A})$ then $\Gamma^d(X/S) = \operatorname{Proj}(\bigoplus_{k\geq 0} \Gamma^d(\mathcal{A}_k))$ in analogy with the description $\operatorname{Sym}^d(X) = \operatorname{Proj}(\bigoplus_{k\geq 0} \operatorname{TS}^d(\mathcal{A}_k))$. The Chow scheme of $X = \operatorname{Proj}(\mathcal{A})$ is defined as the projective scheme corresponding to the subalgebra of $\bigoplus_{k\geq 0} \Gamma^d(\mathcal{A}_k)$ generated by degree one elements. The reduction of this scheme is the classical Chow variety.

3.1. **The symmetric product.**

Definition (3.1.1). Let X be a scheme over S and d a positive integer. We let the symmetric group on d letters $\mathfrak{S}_d$ act by permutations on $(X/S)^d = X \times_S X \times_S \cdots \times_S X$. When X/S is separated, the geometric quotient of $(X/S)^d$ by the action of $\mathfrak{S}_d$ exists as an algebraic space [Ryd07b] and we denote it by $\operatorname{Sym}^d(X/S) := (X/S)^d/\mathfrak{S}_d$. The scheme (or algebraic space) $\operatorname{Sym}^d(X/S)$ is called the d^{th} *symmetric product* of X over S and is also denoted $\operatorname{Symm}^d(X/S)$, $(X/S)^{(d)}$ or $X^{(d)}$ by some authors.

Definition (3.1.2). Let X/S be a scheme. We say that X/S is AF if the following condition is satisfied:

(AF) Every finite set of points $x_1, x_2, \dots, x_n$ over the same point $s \in S$ is contained in an affine open subset of X.

Remark **(3.1.3).** If X has an ample sheaf relative to S, then X/S is AF, cf. [EGA$_{\mathrm{II}}$, Cor. 4.5.4]. It is also clear from [EGA$_{\mathrm{II}}$, Cor. 4.5.4] that if X/S is AF then so is $X \times_S S'/S'$ for any base change $S' \to S$. It can further be seen that if X/S is AF then X/S is separated.

Remark **(3.1.4).** Let X/S be an AF-scheme and let d be a positive integer. By Theorem (2.3.2) the geometric quotient $\mathrm{Sym}^d(X/S)$ is then a scheme. Let (S_α) be an affine cover of S and let $(U_{\alpha\beta})$ be an affine cover of $X \times_S S_\alpha$ such that any set of d points of X lying over the same point $s \in S_\alpha$ is included in some $U_{\alpha\beta}$. Then $(U_{\alpha\beta}/S_\alpha)^d$ is an open cover of $(X/S)^d$ by affine schemes. Thus $\coprod_{\alpha,\beta} \mathrm{Sym}^d(U_{\alpha\beta}/S_\alpha) \to \mathrm{Sym}^d(X/S)$ is an open covering by affines.

In the remainder of this section we will study the symmetric product when $S = \mathrm{Spec}(A)$ is an affine scheme and X/S is projective. We will use the following notation:

Notation (3.1.5). Let A be a ring and let $B = \bigoplus_{k \geq 0} B_k$ be a graded A-algebra finitely generated by elements in degree one. Let $S = \mathrm{Spec}(A)$ and $X = \mathrm{Proj}(B)$ with very ample sheaf $\mathcal{O}_X(1) = \widetilde{B(1)}$ and canonical morphism $q : X \to S$.

Further we let $C = \bigoplus_{k \geq 0} \mathrm{T}^d_A(B_k) \subset \mathrm{T}^d_A(B)$. Then $(X/S)^d = \mathrm{Proj}(C)$ and $\mathrm{Proj}(C) \hookrightarrow \mathbb{P}(C_1) = \mathbb{P}\big(\mathrm{T}^d_A(B_1)\big)$ is the Segre embedding of $(X/S)^d$ corresponding to the embedding $X = \mathrm{Proj}(B) \hookrightarrow \mathbb{P}(B_1)$. The permutation of the factors induces an action of the symmetric group $\mathfrak{S}_d$ on C and we let $D = C^{\mathfrak{S}_d} = \bigoplus_{k \geq 0} \mathrm{TS}^d_A(B_k)$ be the graded invariant ring.

By Proposition (2.3.4) we have that $\mathrm{Sym}^d(X/S) := \mathrm{Proj}(C)/\mathfrak{S}_d = \mathrm{Proj}(D)$. If A is noetherian, then D is finitely generated and $\mathrm{Sym}^d(X/S)$ is a weighted projective space. In general, however, we do not know that D is finitely generated. We do know that $\mathrm{Sym}^d(X/S) \to S$ is universally closed though, as $(X/S)^d \to S$ is projective.

Lemma (3.1.6). *Let $x_1, x_2, \dots, x_d \in X$ be points such that $q(x_1) = q(x_2) = \dots = q(x_d) = s$. Then there exists a positive integer n and an element $f \in B_n \subseteq \Gamma\big(X, \mathcal{O}_X(n)\big)$ such that $x_1, x_2, \dots, x_d \in X_f = \mathrm{D}_+(f)$. If the residue field $k(s)$ has at least d elements then it is possible to take $n = 1$.*

Proof. The existence of f for some n follows from [EGA$_{\mathrm{II}}$, Cor. 4.5.4]. For the last assertion, assume that $k(s)$ has at least d elements. As we can lift any element $\overline{f} \in B_n \otimes_A k(s)$ to an element $f \in B_n$ after multiplying with an invertible element of $k(s)$, we can assume that $A = k(s)$. Replacing B with the symmetric algebra

$\mathrm{S}(B_1) = k[t_0, t_1, \dots, t_r]$ we can further assume that B is a polynomial ring and $X = \mathbb{P}^r_{k(s)}$.

An element of $B_1 = \Gamma\big(X, \mathcal{O}_X(1)\big)$ is then a linear form $f = a_0t_0 + a_1t_1 + \dots + a_rt_r$ with $a_i \in k(s)$ and can be thought of as a $k(s)$-rational point of $(\mathbb{P}^r_{k(s)})^\vee$. The linear forms which are zero in one of the x_i's form a proper closed linear subset of all linear forms. Thus if $k(s)$ is infinite then there is a $k(s)$-rational point corresponding to a linear form non-zero in every x_i. If $k = k(s)$ is finite, then at most $(|k|^r - 1)/|k^*|$ linear forms are zero at a certain x_i and equality holds when x_i is k-rational. Thus at most

$$\begin{aligned} d(|k|^r - 1)/(|k| - 1) &\leq (|k|^{r+1} - |k|)/(|k| - 1) \\ &= (|k|^{r+1} - 1)/(|k| - 1) - 1 \end{aligned}$$

linear forms contain at least one of the $x_1, x_2, \dots, x_d$ and hence there is at least one linear form which does not vanish on any of the points. □

Proposition (3.1.7). *The product $X^d = X \times_S X \times_S \dots \times_S X$ is covered by $\mathfrak{S}_d$-stable affine open subsets of the form $X_f \times_S X_f \times_S \dots \times_S X_f$ where $f \in B_n \subseteq \Gamma\big(X, \mathcal{O}_X(n)\big)$ for some n. If every residue field of S has at least d elements then the open subsets with $f \in B_1 \subseteq \Gamma\big(X, \mathcal{O}_X(1)\big)$ cover X^d.*

Proof. Follows immediately from Lemma (3.1.6). □

Corollary (3.1.8). *The symmetric product $\mathrm{Sym}^d(X/S)$ is covered by open affine subsets $\mathrm{Sym}^d\big(X_f/S\big)$ with $f \in B_n \subseteq \Gamma\big(X, \mathcal{O}_X(n)\big)$ for some n. If every residue field of S has at least d elements then those affine subsets with $n = 1$ cover $\mathrm{Sym}^d(X/S)$.*

Corollary (3.1.9). *The symmetric product $Y = \mathrm{Sym}^d(X/S) = \mathrm{Proj}(D)$ is covered by Y_g where $g \in D_1 \subseteq \Gamma\big(Y, \mathcal{O}_Y(1)\big)$, i.e., $Y = \mathrm{Proj}(D)$ is covered in degree one.*

Proof. Let $A \hookrightarrow A'$ be a finite flat extension such that every residue field of A' has at least d elements, e.g., the extension $A' = A \otimes_{\mathbb{Z}} \Lambda_d$ suffices by Lemma (1.2.3). Let $B' = B \otimes_A A'$ and $C' = C \otimes_A A'$ and let $D' = D \otimes_A A' = \bigoplus_{n \geq 0} \mathrm{TS}^d_A(B_n) \otimes_A A'$. Then $D' = \bigoplus_{n \geq 0} \mathrm{TS}^d_{A'}(B'_n)$ as $A \hookrightarrow A'$ is flat. Note that if $f' \in B'_n$ then $g' = f' \otimes f' \otimes \dots \otimes f' \in D'_n$ and $\mathrm{Sym}^d(X'_{f'}/S) = \mathrm{D}_+(g')$ as open subsets of $\mathrm{Sym}^d(X'/S')$. Thus Corollary (3.1.8) shows that $\sqrt{D'_1 D'_+} = D'_+$. As $\mathrm{Spec}(A') \to \mathrm{Spec}(A)$ is surjective it follows that $\sqrt{D_1 D_+} = D_+$. □

We now use the degree bound on the generators of $\mathrm{TS}^d_A(A[x_1, \dots, x_r])$ obtained in Corollary (1.3.6) to get something very close to a degree bound on the generators of $D = \bigoplus_{k \geq 0} \mathrm{TS}^d_A(B_k)$ when $B = A[x_0, x_1, \dots, x_r]$ is the polynomial ring.

Proposition (3.1.10). *Let N be a positive integer and $D_{\leq N}$ be the subring of $D = \bigoplus_{k \geq 0} \mathrm{TS}^d_A(B_k)$ generated by elements of degree at most N. Then the inclusion $D_{\leq N} \hookrightarrow D$ induces a morphism $\psi_N : \mathrm{Proj}(D) \to \mathrm{Proj}(D_{\leq N})$. Further we have that:*

(i) *If* $B = A[x_0, x_1, \dots, x_r]$ *is a polynomial ring and* $N \geq r(d-1)$ *then* ψ_N *is an isomorphism.*
(ii) *If* A *is purely of characteristic zero, i.e., a* $\mathbb{Q}$*-algebra, then* ψ_N *is an isomorphism for any* N*.*

Proof. By Corollary (3.1.9) the morphism ψ_N is everywhere defined for $N \geq 1$. Let $A \hookrightarrow A'$ be a finite flat extension such that every residue field of A' has at least d elements, e.g., $A' = A \otimes_{\mathbb{Z}} \Lambda_d$ as in Lemma (1.2.3). If we let $C' = C \otimes_A A'$ then we have that $D' = D \otimes_A A' = C'^{\mathfrak{S}_d}$ and $D'_{\leq N} = D_{\leq N} \otimes_A A'$ as $A \hookrightarrow A'$ is flat. If $\psi'_N : \operatorname{Proj}(D') \to \operatorname{Proj}(D'_{\leq N})$ is an isomorphism then so is ψ_N as $A \hookrightarrow A'$ is faithfully flat. Replacing A with A', it is thus enough to prove the corollary when every residue field of S has at least d elements. Hence we can assume that we have a cover of $\operatorname{Proj}(D)$ by $\mathrm{D}_+\big(f^{\otimes d}\big)$ with $f \in B_1$ by Corollary (3.1.8).

We have that $D_{(f^{\otimes d})} = \mathrm{TS}^d_A(B_{(f)})$ and this latter ring is generated by elements of degree $\leq \max\{r(d-1), 1\}$ for arbitrary A and by elements of degree one when A is purely of characteristic zero by Corollary (1.3.6). As noted in Remark (1.3.3) this implies that $D_{(f^{\otimes d})} = (D_{\leq N})_{(f^{\otimes d})}$ which shows (i) and (ii). □

Corollary (3.1.11). *Let* N *be a positive integer and* D_N *be the subring of* $D^{(N)} = \bigoplus_{k\geq 0} \mathrm{TS}^d_A(B_{Nk})$ *generated by* $\mathrm{TS}^d_A(B_N)$*. Then the inclusion* $D_N \hookrightarrow D^{(N)}$ *induces a morphism* $\psi_N : \operatorname{Proj}(D) \to \operatorname{Proj}(D_N)$*. Further we have that:*

(i) *If* $B = A[x_0, x_1, \dots, x_r]$ *is a polynomial ring and* $N \geq r(d-1)$ *then* ψ_N *is an isomorphism.*
(ii) *If* A *is purely of characteristic zero, i.e., a* $\mathbb{Q}$*-algebra, then* ψ_N *is an isomorphism for any* N*.*

Proof. Let $D_{\leq N}$ be the subring of $D = \bigoplus_{k\geq 0} \mathrm{TS}^d_A(B_k)$ generated by elements of degree at most N. As $\operatorname{Proj}(D)$ is covered in degree one by Corollary (3.1.9) then so is $\operatorname{Proj}(D_{\leq N})$. In fact, as $\operatorname{Proj}(D) \to \operatorname{Spec}(A)$ is universally closed, it follows that $\operatorname{Proj}(D) \to \operatorname{Proj}(D_{\leq N})$ is surjective. By Proposition (2.2.7) (iv) it then follows that $D_N \hookrightarrow (D_{\leq N})^{(N)}$ induces an isomorphism $\operatorname{Proj}(D^{(N)}_{\leq N}) \to \operatorname{Proj}(D_N)$. The corollary thus follows from Proposition (3.1.10). □

Theorem (3.1.12). *Let* S *be any scheme and* $\mathcal{E}$ *a quasi-coherent* $\mathcal{O}_S$*-sheaf of finite type. Then for any* $N \geq 1$*, there is a canonical morphism*

$$\operatorname{Sym}^d(\mathbb{P}(\mathcal{E})/S) \to \mathbb{P}\big(\mathrm{TS}^d_{\mathcal{O}_S}(\mathrm{S}^N\mathcal{E})\big).$$

If $\mathcal{L}$ *is a locally free* $\mathcal{O}_S$*-sheaf of constant rank* $r+1$ *then the canonical morphism* $\operatorname{Sym}^d(\mathbb{P}(\mathcal{L})/S) \hookrightarrow \mathbb{P}\big(\mathrm{TS}^d_{\mathcal{O}_S}(\mathrm{S}^N\mathcal{L})\big)$ *is a closed immersion for* $N \geq r(d-1)$*. In particular, it follows that* $\operatorname{Sym}^d(\mathbb{P}(\mathcal{L})/S) \to S$ *is strongly projective.*

Proof. The existence of the morphism follows by Corollary (3.1.11). Part (i) of the same corollary shows that $\operatorname{Sym}^d(\mathbb{P}(\mathcal{L})/S) \hookrightarrow \mathbb{P}\big(\mathrm{TS}^d_{\mathcal{O}_S}(\mathrm{S}^N\mathcal{L})\big)$ is a closed immersion when $N \geq r(d-1)$. As $\mathrm{S}^N\mathcal{L}$ is locally free of constant rank it follows by paragraph (1.1.7) that $\mathrm{TS}^d_{\mathcal{O}_S}(\mathrm{S}^N\mathcal{L})$ is locally free of constant rank which shows that $\mathrm{Sym}^d(\mathbb{P}(\mathcal{L})/S)$ is strongly projective. $\square$

In section 3.3, we will show that the canonical morphism $\mathrm{Sym}^d(\mathbb{P}(\mathcal{E})/S) \to \mathbb{P}\big(\mathrm{TS}^d_{\mathcal{O}_S}(\mathrm{S}^N\mathcal{E})\big)$ is a universal homeomorphism onto its image which is defined to be the Chow scheme $\mathrm{Chow}_{0,d}\big(\mathbb{P}(\mathcal{E})\big)$.

3.2. **The scheme of divided powers.** Let S be any scheme and $\mathcal{A}$ a quasi-coherent sheaf of $\mathcal{O}_S$-algebras. As the construction of $\Gamma^d_A(B)$ commutes with localization with respect to multiplicatively closed subsets of A we may define a quasi-coherent sheaf of $\mathcal{O}_S$-algebras $\Gamma^d_{\mathcal{O}_S}(\mathcal{A})$. We let $\Gamma^d(\mathrm{Spec}(\mathcal{A})/S) = \mathrm{Spec}\big(\Gamma^d_{\mathcal{O}_S}(\mathcal{A})\big)$. The scheme $\Gamma^d(X/S)$ is thus defined for any scheme X affine over S. Similarly we obtain for any homomorphism of quasi-coherent $\mathcal{O}_S$-algebras $\mathcal{A} \to \mathcal{B}$ a morphism of schemes $\Gamma^d(\mathrm{Spec}(\mathcal{B})/S) \to \Gamma^d(\mathrm{Spec}(\mathcal{A})/S)$. This defines a covariant functor $X \mapsto \Gamma^d(X/S)$ from affine schemes over S to affine schemes over S.

It is more difficult to define $\Gamma^d(X/S)$ for any X-scheme S since $\Gamma^d_A(B)$ does not commute with localization with respect to B. In fact, it is not even a B-algebra. In [**I**, 3.1] a certain functor $\underline{\Gamma}^d_{X/S}$ is defined which is represented by $\Gamma^d(X/S)$ when X/S is affine. When X/S is quasi-projective, or more generally an AF-scheme, cf. Definition (3.1.2), then $\underline{\Gamma}^d_{X/S}$ is represented by a scheme [**I**, Thm. 3.1.11]. If X/S is a separated algebraic space, then $\underline{\Gamma}^d_{X/S}$ is represented by a separated algebraic space [**I**, Thm. 3.4.1].

The object representing $\underline{\Gamma}^d_{X/S}$ will be denoted by $\Gamma^d(X/S)$. We briefly state some facts about $\Gamma^d(X/S)$ used in the other sections. We then show that $\Gamma^d(X/S)$ is (quasi-)projective when X/S is (quasi-)projective.

(3.2.1) *The space of divided powers* — For any algebraic scheme X separated above S, there is an algebraic space $\Gamma^d(X/S)$ over S with the following properties:

(i) For any morphism $S' \to S$, there is a canonical base-change isomorphism $\Gamma^d(X/S) \times_S S' \cong \Gamma^d(X \times_S S'/S')$.
(ii) If X/S is an AF-scheme, then $\Gamma^d(X/S)$ is an AF-scheme.
(iii) If $\mathcal{A}$ is a quasi-coherent sheaf on S such that $X = \mathrm{Spec}_S(\mathcal{A})$ is affine S, then $\Gamma^d(X/S) = \mathrm{Spec}_S\big(\Gamma^d_{\mathcal{O}_S}(\mathcal{A})\big)$ is affine over S.
(iv) If $X = \coprod_{i=1}^n X_i$ then $\Gamma^d(X/S)$ is the disjoint union
$$\coprod_{\substack{d_1,d_2,\dots,d_n \geq 0 \\ d_1+d_2+\dots+d_n=d}} \Gamma^{d_1}(X_i/S) \times_S \Gamma^{d_2}(X_2/S) \times_S \dots \times_S \Gamma^{d_n}(X_n/S).$$
(v) If $X \to S$ has one of the properties: finite type, finite presentation, locally of finite type, locally of finite presentation, quasi-compact, finite, integral, flat; then so has $\Gamma^d(X/S) \to S$.

This is Thm. 3.1.11, Prop. 3.1.4, Prop. 3.1.8 and Prop. 4.3.1 of [**I**].

(3.2.2) *Push-forward of cycles* — Let $f : X \to Y$ be any morphism of algebraic schemes separated over S. There is then a natural morphism, push-forward of cycles, $f_* : \Gamma^d(X/S) \to \Gamma^d(Y/S)$ which for affine schemes is given by the covariance of the functor $\Gamma^d_A(\cdot)$. If $f : X \to Y$ is an immersion (resp. a closed immersion, resp. an open immersion) then $f_* : \Gamma^d(X/S) \to \Gamma^d(Y/S)$ is an immersion (resp. closed immersion, resp. open immersion) [**I**, Prop. 3.1.7].

(3.2.3) *Addition of cycles* — Let d, e be positive integers. The composition of the open and closed immersion $\Gamma^d(X/S) \times_S \Gamma^e(X/S) \hookrightarrow \Gamma^{d+e}(X \amalg X/S)$ given by (3.2.1) (iv) and the push-forward $\Gamma^{d+e}(X \amalg X/S) \to \Gamma^{d+e}(X/S)$ along the canonical morphism $X \amalg X \to X$ is called addition of cycles [**I**, Def. 4.1.1].

(3.2.4) *The Sym-Gamma morphism* — Let X/S be a separated algebraic space and let $(X/S)^d = X \times_S X \times_S \cdots \times_S X$. There is an integral surjective morphism $\Psi_X : (X/S)^d \to \Gamma^d(X/S)$, given by addition of cycles, invariant under the permutation of the factors. This gives a factorization $(X/S)^d \to \operatorname{Sym}^d(X/S) \to \Gamma^d(X/S)$ and we denote the second morphism by SG_X [**I**, Prop. 4.1.5].

(3.2.5) *Local description of the scheme of divided powers* — If $U \subseteq X$ is an open subset, then we have already seen that $\Gamma^d(U/S) \subseteq \Gamma^d(X/S)$ is an open subset. Moreover, there is a cartesian diagram

$$\begin{tikzcd}
(U/S)^d \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{Sym}^d(U/S) \arrow[r, "\mathrm{SG}_U"] \arrow[d] \arrow[dr, phantom, "\square"] & \Gamma^d(U/S) \arrow[d] \\
(X/S)^d \arrow[r] & \operatorname{Sym}^d(X/S) \arrow[r, "\mathrm{SG}_X"] & \Gamma^d(X/S).
\end{tikzcd}$$

If X/S is an AF-scheme, then there are Zariski-covers $S = \bigcup S_\alpha$ and $X = \bigcup U_\alpha$ of affine schemes such that $\Gamma^d(X/S) = \bigcup \Gamma^d(U_\alpha/S_\alpha)$. This gives a local description of the SG-map. For an arbitrary separated algebraic space X/S there is a similar étale-local description of SG. If $U \to X$ is an étale morphism, then there is a cartesian diagram

$$\begin{tikzcd}
(U/S)^d|_{\mathrm{fpr}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{Sym}^d(U/S)|_{\mathrm{fpr}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \Gamma^d(U/S)|_{\mathrm{reg}} \arrow[d] \\
(X/S)^d \arrow[r] & \operatorname{Sym}^d(X/S) \arrow[r] & \Gamma^d(X/S)
\end{tikzcd}$$

where the vertical arrows are étale [**I**, Prop. 4.2.4]. Here fpr and reg denotes the open locus where the corresponding maps are *fixed-point reflecting* and *regular*. If $\coprod U_\alpha \to X$ is an étale cover, then $\coprod \Gamma^d(U_\alpha/S)|_{\mathrm{fpr}} \to \Gamma^d(X/S)$ is an étale cover. Thus, we have an étale-local description of SG in affine schemes.

We now give a similar treatment of $\Gamma^d(X/S)$ for $X = \operatorname{Proj}(B)$ projective as that given for $\operatorname{Sym}^d(X/S)$ in the previous section.

Proposition (3.2.6). *Let $S = \operatorname{Spec}(A)$ where A is affine and let $X = \operatorname{Proj}(B)$ where B is a graded A-algebra finitely generated in degree one. Then $\Gamma(X/S)$ is covered by open subsets of the form $\Gamma^d(X_f/S)$ with $f \in B_n$ for some n. If every residue field of S has at least d elements, then is enough to consider open subsets with $n = 1$.*

Proof. Follows from Proposition (3.1.7) using that $\Psi_X((X_f/S)^d) = \Gamma^d(X_f/S)$. $\square$

Proposition (3.2.7). *Let A be a ring and B a graded A-algebra finitely generated in degree one. Let $W = \operatorname{Proj}\bigl(\bigoplus_{k\geq 0} \Gamma^d_A(B_k)\bigr)$. Then W is covered by the open subsets of the form $W_{\gamma^d(b)}$ where $b \in B_n$ for some n. If every residue field of $S = \operatorname{Spec}(A)$ has at least d elements, then is enough to consider open subsets with $b \in B_1$.*

Proof. Let F be a graded flat A-algebra with a surjection $F \twoheadrightarrow B$. Consider the induced surjective homomorphism $\bigoplus_{k\geq 0} \Gamma^d_A(F_k) \twoheadrightarrow \bigoplus_{k\geq 0} \Gamma^d_A(B_k)$ and the corresponding closed immersion $W \hookrightarrow W' = \operatorname{Proj}\bigl(\bigoplus_{k\geq 0} \Gamma^d_A(F_k)\bigr)$. The open subset of $W' = \operatorname{Sym}^d(\operatorname{Proj}(F)/S)$ given by $\gamma^d(f) = 0$, where $f \in F_n$, coincides with the open subset $\operatorname{Sym}^d(\operatorname{Proj}(F)_f/S)$. These subsets cover W' by Corollary (3.1.8). The proposition follows immediately. $\square$

Corollary (3.2.8). *Let S be any scheme and let $\mathcal{A}$ be a graded quasi-coherent $\mathcal{O}_S$-algebra of finite type generated in degree one. Then $\Gamma^d\bigl(\operatorname{Proj}(\mathcal{A})/S\bigr)$ and $W = \operatorname{Proj}\bigl(\bigoplus_{k\geq 0} \Gamma^d_A(\mathcal{A}_k)\bigr)$ are canonically isomorphic. Under this isomorphism, the open subset $\Gamma^d\bigl(\operatorname{Proj}(\mathcal{A})_f\bigr)$ is identified with $W_{\gamma^d(f)}$ for any homogeneous element $f \in \mathcal{A}$.*

Proof. By Propositions (3.2.6) and (3.2.7) the open subsets $\Gamma^d\left(\operatorname{Spec}\left(\mathcal{A}_{(f)}\right)\right)$ and $W_{\gamma^d(f)}$ for $f \in \mathcal{A}_n$, covers $\Gamma^d(\operatorname{Proj}(\mathcal{A}))$ and W respectively. As these subsets are canonically isomorphic the corollary follows. $\square$

Proposition (3.2.9). *Let S be any scheme and let $\mathcal{A}$ be a graded quasi-coherent $\mathcal{O}_S$-algebra of finite type generated in degree one. Let $\mathcal{D} = \bigoplus_{k\geq 0} \Gamma^d_A(\mathcal{A}_k)$. Let N be a positive integer and let $\mathcal{D}_N$ be the subring of $\mathcal{D}^{(N)} = \bigoplus_{k\geq 0} \Gamma^d_A(\mathcal{A}_{Nk})$ generated by $\Gamma^d_A(\mathcal{A}_N)$. The inclusion $\mathcal{D}_N \hookrightarrow \mathcal{D}^{(N)}$ induces a morphism $\psi_N : \operatorname{Proj}(\mathcal{D}) \to \operatorname{Proj}(\mathcal{D}_N)$. Furthermore*

(i) *If $\mathcal{A}$ is locally generated by at most $r+1$ elements and $N \geq r(d-1)$ then ψ_N is an isomorphism.*
(ii) *If S is purely of characteristic zero, i.e., a $\mathbb{Q}$-scheme, then ψ_N is an isomorphism for every N.*

Proof. The statements are local on S so we may assume that $S = \operatorname{Spec}(A)$ is affine and $\mathcal{A} = \widetilde{B}$ where B is a graded A-algebra finitely generated in degree one. Choose a surjection $B' = A[x_0, x_1, \dots, x_r] \twoheadrightarrow B$. Let $D = \bigoplus_{k\geq 0} \Gamma^d_A(B_k)$, $D' = \bigoplus_{k\geq 0} \Gamma^d_A(B'_k)$ and let D_N and D'_N be the subrings of $D^{(N)}$ and $D'^{(N)}$ generated by degree one elements. Then we have a commutative diagram

$$\begin{tikzcd} D'_N \arrow[r, two heads] \arrow[d, hook] \arrow[dr, phantom, "\circ"] & D_N \arrow[d, hook] \\ D'^{(N)} \arrow[r, two heads] & D^{(N)}. \end{tikzcd} \tag{3.2.9.1}$$

By Corollary (3.1.11) the inclusion $D'_N \hookrightarrow D'^{(N)}$ induces a morphism

$$\psi'_N : \operatorname{Proj}\left(D'^{(N)}\right) \to \operatorname{Proj}(D'_N)$$

having properties (i) and (ii). From the commutative diagram (3.2.9.1) it follows that the inclusion $D_N \hookrightarrow D^{(N)}$ induces a morphism $\psi_N : \operatorname{Proj}(D^{(N)}) \to \operatorname{Proj}(D_N)$ with the same properties. □

Theorem (3.2.10). *If $X \to S$ is strongly projective (resp. strongly quasi-projective) then $\Gamma^d(X/S) \to S$ is strongly projective (resp. strongly quasi-projective). If $X \to S$ is projective (resp. quasi-projective) and S is quasi-compact and quasi-separated then $\Gamma^d(X/S) \to S$ is projective (resp. quasi-projective).*

Proof. In the strongly projective (resp. strongly quasi-projective) case we immediately reduce to the case where $X = \mathbb{P}_S(\mathcal{L})$ for some locally free $\mathcal{O}_S$-module $\mathcal{L}$ of finite rank $r+1$, using the push-forward (3.2.2), and the result follows from Theorem (3.1.12).

If S is quasi-compact and quasi-separated and $X \to S$ is projective (resp. quasi-projective) then there is a closed immersion (resp. immersion) $X \hookrightarrow \mathbb{P}_S(\mathcal{E})$ for some quasi-coherent $\mathcal{O}_S$-module $\mathcal{E}$ of finite type. It is enough to show that $\Gamma(\mathbb{P}_S(\mathcal{E}))$ is projective. As $\Gamma(\mathbb{P}_S(\mathcal{E})) = \operatorname{Proj}\bigl(\bigoplus_{k\geq 0} \Gamma^d(\mathrm{S}^k(\mathcal{E})\bigr)$ by Corollary (3.2.8), this follows from Proposition (3.2.9) and the quasi-compactness of S. □

3.3. **The Chow scheme.** Let k be a field and let E be a vector space over k with basis $x_0, x_1, \dots, x_n$. Let $E^\vee$ be the dual vector space with dual basis $y_0, y_1, \dots, y_n$. Let $X = \mathbb{P}(E) = \mathbb{P}^n_k$. If k'/k is a field extension then a point $x : \operatorname{Spec}(k') \to X$ is given by coordinates $(x_0 : x_1 : \dots : x_n)$ in k'. To x we associate the *Chow form* $F_x(y_0, y_1, \dots, y_n) = \sum_{i=0}^n x_i y_i \in k'[y_0, y_1, \dots, y_n]$ which is defined up to a constant.

A zero-cycle on $X = \mathbb{P}^n_k$ is a formal sum of closed points. To any zero-dimensional subscheme $Z \hookrightarrow X$ we associate the zero-cycle $[Z]$ defined as the sum of its points with multiplicities. If $\mathcal{Z} = \sum_j a_j [z_j]$ is a zero-cycle on X and k'/k a field extension then we let $\mathcal{Z}_{k'} = \mathcal{Z} \times_k k' = \sum_j a_j [z_j \times_k k']$. It is clear that if $Z \hookrightarrow X$ is a zero-dimensional subscheme then $[Z] \times_k k' = [Z \times_k k']$.

We say that a cycle is effective if its coefficients are positive. The degree of a cycle $\mathcal{Z} = \sum_j a_j [z_j]$ is defined as $\deg(\mathcal{Z}) = \sum_j a_j \deg\bigl(k(z_j)/k\bigr)$. It is clear that $\deg(\mathcal{Z}_{k'}) = \deg(\mathcal{Z})$ for any field extension k'/k.

Let $\mathcal{Z}$ be an effective zero-cycle on X and choose a field extension k'/k such that $\mathcal{Z}_{k'} = \sum_j a_j [z'_j]$ is a sum of k'-points, i.e., $k(z'_j) = k'$. We then define its Chow form as $F_{\mathcal{Z}} = \prod_j F_{z'_j}^{a_j}$. It is easily seen that

(i) $F_{\mathcal{Z}}$ does not depend on the choice of field extension k'/k.
(ii) $F_{\mathcal{Z}}$ has coefficients in k.
(iii) The degree of $F_{\mathcal{Z}}$ coincides with the degree of $\mathcal{Z}$.
(iv) $\mathcal{Z}$ is determined by $F_{\mathcal{Z}}$.

Further, if k is perfect there is a correspondence between zero-cycles of degree d on X and Chow forms of degree d, i.e., homogeneous polynomials, $F \in k[y_0, y_1, \dots, y_n]$ which splits into d linear forms after a field extension. The Chow forms of degree d with coefficients in k is a subset of the linear forms on $\mathbb{P}\bigl(\mathrm{S}^d(E^\vee)\bigr)$ and thus a subset of the k-points of $\mathbb{P}\bigl(\mathrm{S}^d(E^\vee)^\vee\bigr) = \mathbb{P}\bigl(\mathrm{TS}^d(E)\bigr)$.

(3.3.1) *The Chow variety* — Classically it is shown that for $r \geq 0$ and $d \geq 1$ there is a closed subset of $\mathbb{P}\bigl(\mathrm{T}^{r+1}(\mathrm{TS}^d(E))\bigr)$ parameterizing r-cycles of degree d on $\mathbb{P}(E)$. The Chow variety $\mathrm{Chow}_{r,d}\bigl(\mathbb{P}(E)\bigr)$ is then taken as the *reduced* scheme corresponding to this subset. More generally, if S is any scheme and $\mathcal{L}$ is a locally free sheaf then there is a closed subset of $\mathbb{P}_S\bigl(\mathrm{T}^{r+1}(\mathrm{TS}^d(\mathcal{L}))\bigr)$ parameterizing r-cycles of degree d on $\mathbb{P}_S(\mathcal{L})$.

We will now show that the classical Chow variety parameterizing zero-cycles of degree d has a canonical closed *subscheme* structure. We begin with the case where S is the spectrum of a field.

(3.3.2) *The Chow scheme for* $\mathbb{P}(E)/k$ — Let k'/k be a field extension such that k' is algebraically closed. As $(\mathbb{P}(E)/k)^d \to \mathrm{Sym}^d(\mathbb{P}(E)/k)$ is integral, it is easily seen that a k'-point of $\mathrm{Sym}^d(\mathbb{P}(E)/k)$ corresponds to an unordered tuple $(x_1, x_2, \dots, x_d)$ of k'-points of $\mathbb{P}(E)$. Assigning such a tuple the Chow form of the cycle $[x_1]+[x_2]+\dots+[x_d]$ gives a map $\mathrm{Hom}\bigl(k', \mathrm{Sym}^d(\mathbb{P}(E)/k)\bigr) \to \mathrm{Hom}\bigl(k', \mathbb{P}(\mathrm{TS}^d(E))\bigr)$. It is easily seen to be compatible with the homomorphism of algebras

$$\bigoplus_{k\geq 0} \mathrm{S}^k\bigl(\mathrm{TS}^d(E)\bigr) \to \bigoplus_{k\geq 0} \mathrm{TS}^d\bigl(\mathrm{S}^k(E)\bigr)$$

and thus extends to a morphism of schemes

$$\mathrm{Sym}^d\bigl(\mathbb{P}(E)/k\bigr) \to \mathbb{P}\bigl(\mathrm{TS}^d(E)\bigr).$$

It is further clear that the image of this morphism consists of the Chow forms of degree d and that $\mathrm{Sym}^d\bigl(\mathbb{P}(E)/k\bigr) \to \mathbb{P}\bigl(\mathrm{TS}^d(E)\bigr)$ is universally injective and hence a universal homeomorphism onto its image as $\mathrm{Sym}^d\bigl(\mathbb{P}(E)/k\bigr)$ is projective. We let $\mathrm{Chow}_{0,d}\bigl(\mathbb{P}(E)\bigr)$ be the scheme-theoretical image of this morphism.

More generally, we define $\mathrm{Chow}_{0,d}\bigl(\mathbb{P}(\mathcal{L})/S\bigr)$ for any locally free sheaf $\mathcal{L}$ on S as follows:

Definition-Proposition (3.3.3). *Let S be a scheme and $\mathcal{L}$ a locally free $\mathcal{O}_S$-sheaf of finite type. Then the homomorphism $\bigoplus_{k\geq 0} \mathrm{S}^k\mathrm{TS}^d_{\mathcal{O}_S}(\mathcal{L}) \to \bigoplus_{k\geq 0} \mathrm{TS}^d_{\mathcal{O}_S}(\mathrm{S}^k\mathcal{L})$ induces a morphism*

$$\varphi_{\mathcal{L}} : \mathrm{Sym}^d\bigl(\mathbb{P}(\mathcal{L})/S\bigr) \to \mathbb{P}\bigl(\mathrm{TS}^d_{\mathcal{O}_S}(\mathcal{L})\bigr)$$

which is a universal homeomorphism onto its image. We let $\mathrm{Chow}_{0,d}\bigl(\mathbb{P}(\mathcal{L})\bigr)$ *be its scheme-theoretic image.*

Proof. The question is local so we can assume that $S = \mathrm{Spec}(A)$ and $\mathcal{L} = \widetilde{M}$ where M is a free A-module of finite rank. Corollary (3.1.11), with $N = 1$ and $B = \bigoplus_{k\geq 0} \mathrm{S}^k M$, shows that $\bigoplus_{k\geq 0} \mathrm{S}^k\mathrm{TS}^d_A(M) \to \bigoplus_{k\geq 0} \mathrm{TS}^d_A(\mathrm{S}^k M)$ induces a well-defined morphism $\mathrm{Sym}^d\bigl(\mathbb{P}(\mathcal{L})/S\bigr) \to \mathbb{P}\bigl(\mathrm{TS}^d_{\mathcal{O}_S}(\mathcal{L})\bigr)$.

To show that $\mathrm{Sym}^d\bigl(\mathbb{P}(\mathcal{L})/S\bigr) \to \mathbb{P}\bigl(\mathrm{TS}^d_{\mathcal{O}_S}(\mathcal{L})\bigr)$ is a universal homeomorphism onto its image it is enough to show that it is universally injective as $\mathrm{Sym}^d\bigl(\mathbb{P}(\mathcal{L})/S\bigr) \to S$ is universally closed. As $\mathcal{L}$ is flat over S the symmetric product commutes with base change and it is enough to show that $\mathrm{Sym}^d\bigl(\mathbb{P}(\mathcal{L})/S\bigr) \to \mathbb{P}\bigl(\mathrm{TS}^d_{\mathcal{O}_S}(\mathcal{L})\bigr)$ is injective when S is a field. This was discussed above. $\square$

If $X \hookrightarrow \mathbb{P}(\mathcal{L})$ is a closed immersion (resp. an immersion) then the subset of $\mathrm{Chow}_{0,d}\bigl(\mathbb{P}(\mathcal{L})\bigr)$ parameterizing cycles with support in X is closed (resp. locally closed). In fact, it is the image of the morphism

$$\mathrm{Sym}^d(X/S) \to \mathrm{Sym}^d\bigl(\mathbb{P}(\mathcal{L})/S\bigr) \to \mathrm{Chow}_{0,d}\bigl(\mathbb{P}(\mathcal{L})\bigr). \tag{3.3.3.1}$$

As $\mathrm{Sym}^d\bigl(\mathbb{P}(\mathcal{L})/S\bigr) = \Gamma^d\bigl(\mathbb{P}(\mathcal{L})/S\bigr)$, this morphism factors through $\mathrm{Sym}^d(X/S) \to \Gamma^d(X/S)$. Moreover, as $\mathrm{Sym}^d(X/S) \to \Gamma^d(X/S)$ is a homeomorphism [**I**, Cor. 4.2.5], the morphism

$$\Gamma^d(X/S) \to \mathrm{Sym}^d\bigl(\mathbb{P}(\mathcal{L})/S\bigr) \to \mathrm{Chow}_{0,d}\bigl(\mathbb{P}(\mathcal{L})\bigr). \tag{3.3.3.2}$$

has the same image as (3.3.3.1). Since Γ^d is more well-behaved, e.g., commutes with base change $S' \to S$, the following definition is reasonable:

Definition (3.3.4). Let S be any scheme and $\mathcal{L}$ a locally free sheaf on S. If $X \hookrightarrow \mathbb{P}(\mathcal{L})$ is a closed immersion we let $\mathrm{Chow}_{0,d}\bigl(X \hookrightarrow \mathbb{P}(\mathcal{L})\bigr)$ be the scheme-theoretic image of $\Gamma^d(X/S) \hookrightarrow \Gamma^d\bigl(\mathbb{P}(\mathcal{L})/S\bigr) \to \mathrm{Chow}_{0,d}\bigl(\mathbb{P}(\mathcal{L})\bigr)$. If $X \hookrightarrow \mathbb{P}(\mathcal{L})$ is an immersion we let $\mathrm{Chow}_{0,d}\bigl(X \hookrightarrow \mathbb{P}(\mathcal{L})\bigr)$ be the open subscheme of $\mathrm{Chow}_{0,d}\bigl(\overline{X} \hookrightarrow \mathbb{P}(\mathcal{L})\bigr)$ corresponding to cycles with support in X.

Remark **(3.3.5).** Classically $\mathrm{Chow}_{0,d}\bigl(X \hookrightarrow \mathbb{P}(\mathcal{L})\bigr)$ is defined as the reduced subscheme of $\mathrm{Chow}_{0,d}\bigl(\mathbb{P}(\mathcal{L})\bigr) \hookrightarrow \mathbb{P}\bigl(\mathrm{TS}^d(\mathcal{L})\bigr)$ parameterizing zero-cycles of degree d with support in X. It is clear that this is the reduction of the scheme $\mathrm{Chow}_{0,d}\bigl(X \hookrightarrow \mathbb{P}(\mathcal{L})\bigr)$ as defined in Definition (3.3.4).

Remark **(3.3.6).** If $\mathcal{L}$ is a locally free sheaf on S of finite type then by definition $\mathrm{Chow}_{0,d}\bigl(\mathbb{P}(\mathcal{L})\bigr)$ is $\mathrm{Proj}(\mathcal{B})$ where $\mathcal{B}$ is the image of

$$\bigoplus_{k\geq 0} \mathrm{S}^k\bigl(\mathrm{TS}^d(\mathcal{L})\bigr) \to \bigoplus_{k\geq 0} \mathrm{TS}^d\bigl(\mathrm{S}^k(\mathcal{L})\bigr)$$

i.e., $\mathcal{B}$ is the subalgebra of $\bigoplus_{k\geq 0} \mathrm{TS}^d\bigl(\mathrm{S}^k(\mathcal{L})\bigr)$ generated by degree one elements. If $X \hookrightarrow \mathbb{P}(\mathcal{L})$ is a closed immersion then $X = \mathrm{Proj}(\mathcal{A})$ where $\mathcal{A}$ is a quotient of $\mathrm{S}(\mathcal{L})$.

The Chow scheme $\mathrm{Chow}_{0,d}\bigl(X \hookrightarrow \mathbb{P}(\mathcal{L})\bigr)$ is then $\mathrm{Proj}(\mathcal{B})$ where $\mathcal{B}$ is the subalgebra of $\bigoplus_{k\geq 0} \Gamma^d_{\mathcal{O}_S}(\mathcal{A}_k)$ generated by degree one elements, cf. Corollary (3.2.8).

Proposition (3.3.7). *Let S be any scheme and let $\mathcal{B}$ be a graded quasi-coherent $\mathcal{O}_S$-algebra of finite type generated in degree one. Then there is a canonical morphism*

$$\varphi_{\mathcal{B}} : \Gamma^d\bigl(\mathrm{Proj}(\mathcal{B})/S\bigr) \to \mathbb{P}\bigl(\Gamma^d_{\mathcal{O}_S}(\mathcal{B}_1)\bigr)$$

which is a universal homeomorphism onto its image. This morphism commutes with base change $S' \to S$ and surjections $\mathcal{B} \twoheadrightarrow \mathcal{B}'$.

Proof. The existence of the morphism follows from Proposition (3.2.9). That $\varphi_{\mathcal{B}}$ is universally injective can be checked on the fibers and this is done in the beginning of this section. The last statements follows from the corresponding statements of the algebra of divided powers. $\square$

Remark (3.3.6) and Proposition (3.3.7) shows that there is a natural extension of the definition of $\mathrm{Chow}_{0,d}\bigl(X \hookrightarrow \mathbb{P}_S(\mathcal{L})\bigr)$ which includes the case where $\mathcal{L}$ need not be locally free. In particular, we obtain a definition valid for arbitrary projective schemes:

Definition (3.3.8). Let X/S be quasi-projective morphism of schemes and let $X \hookrightarrow \mathbb{P}_S(\mathcal{E})$ be an immersion for some quasi-coherent $\mathcal{O}_S$-module $\mathcal{E}$ of finite type. Let $\overline{X}$ be the scheme-theoretic image of X in $\mathbb{P}_S(\mathcal{E})$ which can be written as $\overline{X} = \mathrm{Proj}(\mathcal{B})$ where $\mathcal{B}$ is a quotient of $\mathrm{S}(\mathcal{E})$. We let $\mathrm{Chow}_{0,d}\bigl(\overline{X} \hookrightarrow \mathbb{P}_S(\mathcal{E})\bigr)$ be the scheme-theoretic image of $\varphi_{\mathcal{B}} : \Gamma^d\bigl(\mathrm{Proj}(\mathcal{B})/S\bigr) \to \mathbb{P}\bigl(\Gamma^d_{\mathcal{O}_S}(\mathcal{B}_1)\bigr)$ or equivalently, the scheme-theoretic image of

$$\varphi_{\overline{X},\mathcal{E}} : \Gamma^d\bigl(\mathrm{Proj}(\mathcal{B})/S\bigr) \to \mathbb{P}\bigl(\Gamma^d_{\mathcal{O}_S}(\mathcal{B}_1)\bigr) \hookrightarrow \mathbb{P}\bigl(\Gamma^d_{\mathcal{O}_S}(\mathcal{E})\bigr).$$

We let $\mathrm{Chow}_{0,d}\bigl(X \hookrightarrow \mathbb{P}_S(\mathcal{E})\bigr)$ be the open subscheme of $\mathrm{Chow}_{0,d}\bigl(\overline{X} \hookrightarrow \mathbb{P}_S(\mathcal{E})\bigr)$ given by the image of

$$\Gamma^d(X/S) \subseteq \Gamma^d(\overline{X}/S) \to \mathrm{Chow}_{0,d}\bigl(\overline{X} \hookrightarrow \mathbb{P}_S(\mathcal{E})\bigr).$$

This is indeed an open subscheme as $\Gamma^d\bigl(\overline{X}/S\bigr) \to \mathrm{Chow}_{0,d}\bigl(\overline{X} \hookrightarrow \mathbb{P}(\mathcal{E})\bigr)$ is a homeomorphism by Corollary (3.3.7).

Remark **(3.3.9).** Let S be any scheme, $\mathcal{E}$ a quasi-coherent $\mathcal{O}_S$-module and $X \hookrightarrow \mathbb{P}(\mathcal{E})$ an immersion. Let $S' \to S$ be any morphism and let $X' = X \times_S S'$ and $\mathcal{E}' = \mathcal{E} \otimes_{\mathcal{O}_S} \mathcal{O}_{S'}$. There is a commutative diagram

$$\begin{tikzcd}
\Gamma^d(X'/S') \arrow[r, "\varphi_{X',\mathcal{E}'}"] \arrow[d, "\cong"] \arrow[dr, phantom, "\circ"] & \mathbb{P}_{S'}(\mathcal{E}') \arrow[d, "\cong"] \\
\Gamma^d(X/S) \times_S S' \arrow[r, "\varphi_{X,\mathcal{E}} \times_S \mathrm{id}_{S'}"] & \mathbb{P}_S(\mathcal{E}) \times_S S'
\end{tikzcd}$$

i.e., $\varphi_{X',\mathcal{E}'} = \varphi_{X,\mathcal{E}} \times_S \mathrm{id}_{S'}$. As the underlying sets of $\mathrm{Chow}_{0,d}\bigl(X \hookrightarrow \mathbb{P}(\mathcal{E})\bigr)$ and $\mathrm{Chow}_{0,d}\bigl(X' \hookrightarrow \mathbb{P}(\mathcal{E}')\bigr)$ are the images of $\varphi_{X,\mathcal{E}}$ and $\varphi_{X',\mathcal{E}'}$ it follows that the canonical morphism

$$\mathrm{Chow}_{0,d}\bigl(X' \hookrightarrow \mathbb{P}_{S'}(\mathcal{E}')\bigr) \hookrightarrow \mathrm{Chow}_{0,d}\bigl(X \hookrightarrow \mathbb{P}_S(\mathcal{E})\bigr) \times_S S' \tag{3.3.9.1}$$

is a nil-immersion, i.e., a bijective closed immersion. As the scheme-theoretic image commutes with flat base change [$\mathrm{EGA_{IV}}$, Lem. 2.3.1] the morphism (3.3.9.1) is an isomorphism if $S' \to S$ is flat.

If $Z \hookrightarrow X$ is an immersion (resp. a closed immersion, resp. an open immersion) then there is an immersion (resp. a closed immersion, resp. an open immersion)

$$\mathrm{Chow}_{0,d}\bigl(Z \hookrightarrow \mathbb{P}_S(\mathcal{E})\bigr) \hookrightarrow \mathrm{Chow}_{0,d}\bigl(X \hookrightarrow \mathbb{P}_S(\mathcal{E})\bigr).$$

Proposition (3.3.10). *Let $S = \mathrm{Spec}(A)$ where A is affine and such that every residue field of S has at least d elements. Let $X = \mathrm{Proj}(B)$ where B is a graded A-algebra finitely generated in degree one. Let $D = \bigoplus_{k\geq 0} \Gamma^d_A(B_k)$ and let $E \hookrightarrow D$ be the subalgebra generated by elements of degree one. Then $\mathrm{Chow}\bigl(X \hookrightarrow \mathbb{P}(B_1)\bigr) = \mathrm{Proj}(E)$ is covered by open subsets of the form $\mathrm{Spec}(E_{\gamma^d(f)})$ with $f \in B_1$. Furthermore, $E_{\gamma^d(f)}$ is the subalgebra of $\Gamma^d\bigl(B_{(f)}\bigr)$ generated by elements of degree one, i.e., elements of the form $\times_{i=1}^n \gamma^{d_i}(b_i/f)$ with $b_i \in B_1$.*

Proof. The first statement follows immediately from Proposition (3.2.7) taking into account that the inclusion $E \hookrightarrow D$ induces a surjective morphism $\mathrm{Proj}(D) \to \mathrm{Proj}(E)$ by Proposition (3.3.7). The last statement is obvious. □

4. The relations between the parameter spaces

In this section we show that the morphisms

$$\mathrm{Sym}^d(X/S) \to \Gamma^d(X/S) \to \mathrm{Chow}_{0,d}\bigl(X \hookrightarrow \mathbb{P}(\mathcal{E})\bigr)$$

are universal homeomorphisms with trivial residue field extensions. That the first morphism is a universal homeomorphisms with trivial residue field extensions is shown in [**I**]. We also briefly mention the construction of the morphism $\mathrm{Hilb}^d(X/S) \to \mathrm{Sym}^d(X/S)$.

4.1. The Sym-Gamma morphism.

In this section we discuss some properties of the canonical morphism $\mathrm{SG}_X : \mathrm{Sym}^d(X/S) \to \Gamma^d(X/S)$ defined in (3.2.4). Recall the following basic result:

Proposition (4.1.1). [**I**, Cor. 4.2.5] *Let X/S be a separated algebraic space. The canonical morphism $\mathrm{SG}_X : \mathrm{Sym}^d(X/S) \to \Gamma^d(X/S)$ is a universal homeomorphism with trivial residue field extensions. If S is purely of characteristic zero or X/S is flat, then SG_X is an isomorphism.*

From Proposition (4.1.1) we obtain the following results which only concerns $\mathrm{Sym}^d(X/S)$ but relies on the existence of the well-behaved functor Γ^d and the morphism $\mathrm{Sym}^d(X/S) \to \Gamma^d(X/S)$.

Corollary (4.1.2). *Let $S \to S'$ be a morphism of schemes and X/S a separated algebraic space. The induced morphism $\mathrm{Sym}^d(X'/S') \to \mathrm{Sym}^d(X/S) \times_S S'$ is a universal homeomorphism with trivial residue field extensions. If S' is of characteristic zero then this morphism is an isomorphism. If X'/S' is flat then the morphism is a nil-immersion.*

Proof. Follows from Proposition (4.1.1) and the commutative diagram

$$\begin{tikzcd} \mathrm{Sym}^d(X'/S') \arrow[r] \arrow[d] \arrow[dr, phantom, "\circ"] & \mathrm{Sym}^d(X/S) \times_S S' \arrow[d] \\ \Gamma^d(X'/S') \arrow[r, "\cong"] & \Gamma^d(X/S) \times_S S'. \end{tikzcd}$$

□

Corollary (4.1.3). *Let X/S be a separated algebraic space and $Z \hookrightarrow X$ a closed subscheme. Let $q : (X/S)^d \to \mathrm{Sym}^d(X/S)$ be the quotient morphism. The induced morphism $\mathrm{Sym}^d(Z/S) \to q\big((Z/S)^d\big)$ is a universal homeomorphism with trivial residue field extensions. If S is of characteristic zero then this morphism is an isomorphism. If Z/S is flat then the morphism is a nil-immersion.*

Proof. Follows from Proposition (4.1.1) and the commutative diagram

$$\begin{tikzcd} \mathrm{Sym}^d(Z/S) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circ"] & \mathrm{Sym}^d(X/S) \arrow[d] \\ \Gamma^d(Z/S) \arrow[r, hook] & \Gamma^d(X/S). \end{tikzcd}$$

□

Let us also mention the following result.

Theorem (4.1.4). *Let A be any ring, let B be an A-algebra and let d be a positive integer. Let $\varphi : \Gamma^d_A(B) \to \mathrm{TS}^d_A(B)$ be the canonical homomorphism. Then*

(i) *If $x \in \ker(\varphi)$ then $d!x = 0$ and $x^{d!} = 0$.*
(ii) *If $y \in \mathrm{TS}^d_A(B)$ then $d!y \in \mathrm{im}(\varphi)$ and $y^{d!} \in \mathrm{im}(\varphi)$.*

Proof. This is (1.1.9) and [**II**, Cor. 4.7]. □

It is not difficult to prove that if every prime but p is invertible in A, then $d!$ in the theorem can be replaced with the highest power of p dividing $d!$.

Examples (4.1.5). The following examples are due to C. Lundkvist [Lun08]:

(i) An A-algebra B such that $\Gamma^d_A(B) \to \mathrm{TS}^d_A(B)$ is not injective
(ii) An A-algebra B such that $\Gamma^d_A(B) \to \mathrm{TS}^d_A(B)$ is not surjective
(iii) A surjection $B \to C$ of A-algebras such that $\mathrm{TS}^d_A(B) \to \mathrm{TS}^d_A(C)$ is not surjective
(iv) An A-algebra B such that $\Gamma^d_A(B)_{\mathrm{red}} \hookrightarrow \mathrm{TS}^d_A(B)_{\mathrm{red}}$ is not an isomorphism

(v) An A-algebra B and a base change $A \to A'$ such that the canonical homomorphism $\mathrm{TS}^d_A(B) \otimes_A A' \to \mathrm{TS}^d_{A'}(B')$ is not injective.
(vi) An A-algebra B and a base change $A \to A'$ such that the canonical homomorphism $\mathrm{TS}^d_A(B) \otimes_A A' \to \mathrm{TS}^d_{A'}(B')$ is not surjective.

Remark **(4.1.6).** The *seminormalization* of a scheme X is a universal homemomorphism with trivial residue fields $X^{\mathrm{sn}} \to X$ such that any universal homeomorphism with trivial residue field $X' \to X$ factors uniquely through $X^{\mathrm{sn}} \to X$ [Swa80]. If $X^{\mathrm{sn}} = X$ then we say that X is seminormal. If $X \to Y$ is a morphism and X is seminormal then $X \to Y$ factors canonically through $Y^{\mathrm{sn}} \to Y$.

Using Proposition (4.1.1) it can be shown that $\mathrm{Sym}^d(X/S)^{\mathrm{sn}} = \mathrm{Sym}^d(X^{\mathrm{sn}}/S^{\mathrm{sn}})^{\mathrm{sn}}$. Corollaries (4.1.2) and (4.1.3) then show that in the fibered category of seminormal schemes $\mathbf{Sch}^{\mathrm{sn}}$, taking symmetric products commutes with arbitrary base change and closed subschemes. This is a special property for Sym^d which does not hold for arbitrary quotients.

4.2. **The Gamma-Chow morphism.** Let us first restate the contents of Proposition (3.2.9) taking into account the definition of $\mathrm{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}(\mathcal{E})\big)$.

Proposition (4.2.1). *Let S be a scheme, $q : X \to S$ quasi-projective and $\mathcal{E}$ a quasi-coherent $\mathcal{O}_S$-module of finite type such that there is an immersion $X \hookrightarrow \mathbb{P}(\mathcal{E})$. Let $k \geq 1$ be an integer. Then*

(i) *The canonical map*
$$\mathrm{S}\big(\Gamma^d_{\mathcal{O}_S}(\mathrm{S}^k\mathcal{E})\big) \to \bigoplus_{i\geq 0} \Gamma^d_{\mathcal{O}_S}(\mathrm{S}^{ki}\mathcal{E})$$
induces a morphism
$$\varphi_{\mathcal{E},k} : \Gamma^d(X/S) \hookrightarrow \Gamma^d\big(\mathbb{P}(\mathcal{E})/S\big) \to \mathbb{P}\big(\Gamma^d_{\mathcal{O}_S}(\mathrm{S}^k\mathcal{E})\big)$$
which is a universal homeomorphism onto its image. The scheme-theoretical image of $\varphi_{\mathcal{E},k}$ is by definition $\mathrm{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}(\mathcal{E}^{\otimes k})\big)$.

(ii) *Assume that either $\mathcal{E}$ is locally generated by at most $r+1$ elements and $k \geq r(d-1)$ or S has pure characteristic zero, i.e., is a $\mathbb{Q}$-scheme. Then $\varphi_{\mathcal{E},k}$ is a closed immersion and $\Gamma^d(X/S) \to \mathrm{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}(\mathcal{E}^{\otimes k})\big)$ is an isomorphism.*

Remark **(4.2.2).** As $\Gamma^d(X/S) \to \mathrm{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}(\mathcal{E})\big)$ is a universal homeomorphism, the *topology* of the Chow scheme does not depend on the chosen embedding $X \hookrightarrow \mathbb{P}(\mathcal{E})$.

In higher dimension, it is well-known that the Chow variety $\mathrm{Chow}_{r,d}\big(X \hookrightarrow \mathbb{P}(\mathcal{E})\big)$ does not depend on the embedding $X \hookrightarrow \mathbb{P}(\mathcal{E})$ as a *set*. This follows from the fact that a geometric point corresponds to an r-cycle of degree d [Sam55, §9.4d,h]. The invariance of the topology is also well-known, cf. [Sam55, §9.7]. This implies that the *weak normalization* of the Chow variety does not depend on the embedding in the analytic case, cf. [AN67]. This also follows from functorial descriptions of the

Chow variety over weakly normal schemes as in [Gue96] over $\mathbb{C}$ or more generally in [Kol96, §1.3]. We will now show that the residue fields of $\mathrm{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}(\mathcal{E})\big)$ do not depend on the embedding.

Proposition (4.2.3). *Let S, $q : X \to S$ and $\mathbb{P}(\mathcal{E})$ as in Proposition* (4.2.1). *The morphism $\varphi_{\mathcal{E}} : \Gamma^d(X/S) \to \mathrm{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}(\mathcal{E})\big)$ is a universal homeomorphism with trivial residue field extensions.*

Proof. We have already seen that the morphism $\varphi_{\mathcal{E}}$ is a universal homeomorphism. It is thus enough to show that it has trivial residue field extensions. To show this it is enough to show that for every point $a : \mathrm{Spec}(k) \to \mathrm{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}(\mathcal{E})\big)$ with $k = k^{\mathrm{sep}}$ there exists a, necessarily unique, point $b : \mathrm{Spec}(k) \to \Gamma^d(X/S)$ lifting a, i.e., the diagram

$$\begin{tikzcd} \Gamma^d(X/S) \arrow[r, "\varphi_{\mathcal{E}}"] & \mathrm{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}(\mathcal{E})\big) \\ & \mathrm{Spec}(k) \arrow[u, "a"'] \arrow[ul, dotted, "b"] \end{tikzcd}$$

has a unique filling. By (3.2.1) (i) and Remark (3.3.9) the schemes $\Gamma^d(X/S)$ and $\big(\mathrm{Chow}_{0,d}(X \hookrightarrow \mathbb{P}(\mathcal{E}))\big)_{\mathrm{red}}$ commute with base change, i.e.,

$$\Gamma^d(X/S) \times_S S' = \Gamma^d(X \times_S S'/S')$$
$$\big(\mathrm{Chow}_{0,d}(X \hookrightarrow \mathbb{P}(\mathcal{E})) \times_S S'\big)_{\mathrm{red}} = \mathrm{Chow}_{0,d}\big(X \times_S S' \hookrightarrow \mathbb{P}(\mathcal{E} \otimes_{\mathcal{O}_S} \mathcal{O}_{S'})\big)_{\mathrm{red}}$$

for any $S' \to S$. We can thus assume that $S = \mathrm{Spec}(k)$ and hence that the image of a is a closed point.

Let $r+1$ be the rank of $\mathcal{E}$. The point a then corresponds to a Chow form $F_a \in k[y_0, y_1, \dots, y_r]$ which is homogeneous of degree d. Over $\overline{k} = k^{p^{-\infty}}$ this form factors into linear forms

$$F_a = F_1^{d_1} F_2^{d_2} \dots F_n^{d_n}$$

where $d = d_1 + d_2 + \dots + d_n$. Let $F_j = \sum_j x_i^{(j)} y_i$ and let $k\big(x^{(j)}\big) = k\big(x_0^{(j)}, \dots, k_r^{(j)}\big)$. If we let $d = p^e m$ such that $p \nmid m$, then $k\big(x^{(j)}\big)^{p^e} \subseteq k$ as $F_i^{d_i}$ is k-rational. Thus the exponent of $k\big(x^{(j)}\big)/k$ is at most p^e and it follows by [**II**, Prop. 7.6] that $\Gamma^{d_j}(X/S)$ has a unique k-point b_j corresponding to F_j. The k-point $b = b_1 + b_2 + \dots + b_n$ is a lifting of a. □

Remark **(4.2.4).** Proposition (4.2.3) also follows from the following fact. Let k be a field, E a k-vector space and $X \hookrightarrow \mathbb{P}_k(E)$ a subscheme. Let $\mathcal{Z}$ be an r-cycle on X. The residue field of the point corresponding to $\mathcal{Z}$ in the Chow variety $\mathrm{Chow}_{r,d}\big(X \hookrightarrow \mathbb{P}(E)\big)$, the *Chow field* of $\mathcal{Z}$, does not depend on the embedding $X \hookrightarrow \mathbb{P}_k^n$ [Kol96, Prop-Def. I.4.4].

As $\varphi_{\mathcal{E}^{\otimes k}} : \Gamma^d(X/S) \to \mathrm{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}(\mathcal{E}^{\otimes k})\big)$ is an isomorphism for sufficiently large k by Proposition (4.2.1) the Chow field coincides with the corresponding residue field of $\Gamma^d(X/S)$.

4.3. **The Hilb-Sym morphism.** The Hilbert-Chow morphism can be constructed in several different ways. Mumford [GIT, Ch. 5 §4] constructs a morphism

$$\mathrm{Hilb}^d(\mathbb{P}^n) \to \mathrm{Div}^d((\mathbb{P}^n)^\vee) \cong \mathbb{P}(\mathcal{O}_{(\mathbb{P}^n)^\vee}(d))$$

from the Hilbert scheme of d points to Cartier divisors of degree d on the dual space. This construction is a generalization of the construction of the Chow variety and it follows immediately that the image of this morphism is the Chow variety of $\mathbb{P}^n$. As the Chow variety $\mathrm{Chow}_{0,d}(\mathbb{P}^n)$ does not coincide with $\mathrm{Sym}^d(\mathbb{P}^n) = \Gamma^d(\mathbb{P}^n)$ in general, it is not clear that this construction lifts to a morphism to $\mathrm{Sym}^d(\mathbb{P}^n)$ or to $\Gamma^d(\mathbb{P}^n)$. Neeman solved this [Nee91] constructing a morphism $\mathrm{Hilb}^d(\mathbb{P}^n) \to \mathrm{Sym}^d(\mathbb{P}^n)$ directly without using Chow forms.

There is a natural way of constructing the "Hilbert-Chow" morphism due to Grothendieck [FGA] and Deligne [Del73]. There is a natural map $\mathrm{Hilb}^d(X/S) \to \Gamma^d(X/S)$ taking a flat family $Z \to T$ to its *norm family* [**II**]. We will call this map the *Grothendieck-Deligne norm map* and denote it with HG_X. Using that the symmetric product coincide with the space of divided powers for X/S flat, it follows by functoriality that the morphism $\mathrm{Hilb}^d(X/S) \to \Gamma^d(X/S)$ factors through the symmetric product. To be precise, we have the following natural transformation:

Definition (4.3.1). Let X/S be a separated algebraic space. We let $\mathrm{HS}_X : \mathrm{Hilb}^d(X/S) \to \mathrm{Sym}^d(X/S)$ be the following morphism. Let T be an S-scheme and $f : T \to \mathrm{Hilb}^d(X/S)$ a T-point. Then f corresponds to a subscheme $Z \hookrightarrow X \times_S T$ which is flat and finite over T. There is a commutative diagram

$$\begin{tikzcd}
T \arrow[r] \arrow[dr, "{(f,\mathrm{id}_T)}"'] & \mathrm{Hilb}^d(Z/T) \arrow[r, "\mathrm{HG}_Z"] \arrow[d, hook] & \Gamma^d(Z/T) \arrow[d, hook] & \mathrm{Sym}^d(Z/T) \arrow[l, "\mathrm{SG}_Z", "\cong"'] \arrow[d] \\
 & \mathrm{Hilb}^d(X/S) \times_S T \arrow[r, "\mathrm{HG}_X"] & \Gamma^d(X/S) \times_S T & \mathrm{Sym}^d(X/S) \times_S T \arrow[l, "\mathrm{SG}_X"']
\end{tikzcd}$$

and we let $\mathrm{HS}_X(f)$ be the composition $T \to \mathrm{Sym}^d(X/S) \times_S T \to \mathrm{Sym}^d(X/S)$.

The morphisms HS_X and HG_X are isomorphisms when X/S is a smooth curve [**I**]. They are also both isomorphisms over the non-degeneracy locus as shown in the next section. In [ES04, RS07], it is shown that the closure of the non-degeneracy locus of $\mathrm{Hilb}^d(X/S)$ — the good component — is a blow-up of either $\Gamma^d(X/S)$ or $\mathrm{Sym}^d(X/S)$.

5. Outside the degeneracy locus

In this section we will prove that the morphisms

$$\mathrm{Hilb}^d(X/S) \to \mathrm{Sym}^d(X/S) \to \Gamma^d(X/S) \to \mathrm{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}(\mathcal{E})\big)$$

are all isomorphisms over the open subset parameterizing “non-degenerated families” of points. That the morphism $\mathrm{HG}_X : \mathrm{Hilb}^d(X/S) \to \Gamma^d(X/S)$ is an isomorphism over the non-degeneracy locus is shown in [**II**]. It is thus enough to show that the last two morphisms are isomorphisms outside the degeneracy locus.

5.1. **Families of cycles.** Let $\overline{k}$ be an algebraically closed field and fix a geometric point $s : \mathrm{Spec}(\overline{k}) \to S$. Let $\alpha : \mathrm{Spec}(\overline{k}) \to \mathrm{Sym}^d(X/S)$ be a geometric point above s. As $(X/S)^d \to \mathrm{Sym}^d(X/S)$ is integral, we have that α lifts (non-uniquely) to a geometric point $\beta : \mathrm{Spec}(\overline{k}) \to (X/S)^d$. Let $\pi_i : (X/S)^d \to X$ be the i^{th} projection and let $x_i = \pi_i \circ \beta$. It is easily seen that the different liftings β of α corresponds to the permutations of the d geometric points $x_i : \mathrm{Spec}(\overline{k}) \to X$. This gives a correspondence between $\overline{k}$-points of $\mathrm{Sym}^d(X/S)$ and effective zero-cycles of degree d on X_s.

As $\mathrm{Sym}^d(X/S) \to \Gamma^d(X/S) \to \mathrm{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}(\mathcal{E})\big)$ are universal homeomorphisms, there is a bijection between their geometric points. It is thus reasonable to say that $\mathrm{Sym}^d(X/S)$, $\Gamma^d(X/S)$ and $\mathrm{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}(\mathcal{E})\big)$ parameterize effective zero-cycles of degree d. Moreover, as $\mathrm{Sym}^d(X/S) \to \Gamma^d(X/S) \to \mathrm{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}(\mathcal{E})\big)$ have trivial residue field extensions, there is a bijection between k-points for any field k.

Definition (5.1.1). Let X, S, $\overline{k}$ and s be as above. Let $\mathcal{Z}$ be an effective zero-cycle of degree d on X_s. The residue field of the corresponding point in $\mathrm{Sym}^d(X/S)$, $\Gamma^d(X/S)$ or $\mathrm{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}(\mathcal{E})\big)$ is called the *Chow field* of $\mathcal{Z}$.

Definition (5.1.2). Let k be a field and X a scheme over k. Let k'/k and k''/k be field extensions of k. Two cycles $\mathcal{Z}'$ and $\mathcal{Z}''$ on $X \times_k k'$ and $X \times_k k''$ respectively, are said to be equivalent if there is a common field extension K/k of k' and k'' such that $\mathcal{Z}' \times_{k'} K = \mathcal{Z}'' \times_{k''} K$. If $\mathcal{Z}'$ is a cycle on $X \times_S k'$ equivalent to a cycle on $X \times_S k''$ then we say that $\mathcal{Z}'$ is *defined over* k''.

Remark **(5.1.3).** If $\mathcal{Z}$ is a cycle on $X \times_S k$ then the corresponding morphism $\mathrm{Spec}(\overline{k}) \to \mathrm{Sym}^d(X/S)$ factors through $\mathrm{Spec}(\overline{k}) \to \mathrm{Spec}(k)$. Thus if $\mathcal{Z}$ is defined over a field K then the Chow field is contained in K. Conversely it can be shown that $\mathcal{Z}$ is defined over an inseparable extension of the Chow field. Thus, in characteristic zero the Chow field of $\mathcal{Z}$ is the unique minimal field of definition of $\mathcal{Z}$. In positive characteristic, it can be shown that the Chow field of $\mathcal{Z}$ is the intersection of all minimal fields of definitions of $\mathcal{Z}$, cf. [Kol96, Thm. I.4.5] and [**II**, Prop. 7.13].

Let T be any scheme and $f : T \to \mathrm{Sym}^d(X/S)$, $f : T \to \Gamma^d(X/S)$ or $f : T \to \mathrm{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}(\mathcal{E})\big)$ a morphism. A geometric k-point of T then corresponds to a zero-cycle of degree d on $X \times_S k$. The following definition is therefore natural.

Definition (5.1.4). A family of cycles parameterized by T is a T-point of either $\mathrm{Sym}^d(X/S)$, $\Gamma^d(X/S)$ or $\mathrm{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}(\mathcal{E})\big)$. We use the notation $\mathcal{Z} \to T$ to denote a family of cycles parameterized by T and let $\mathscr{Z}_t$ be the cycle over t, i.e., the cycle corresponding to $\overline{k(t)} \to T \to \mathrm{Sym}^d(X/S)$, etc.

As $\Gamma^d(X/S)$ commutes with base change and has other good properties it is the "correct" parameter scheme and the morphisms $T \to \Gamma^d(X/S)$ are the "correct" families of cycles.

5.2. **Non-degenerated families.**

(5.2.1) *Non-degenerate families of subschemes* — Let k be a field and X be a k-scheme. If $Z \hookrightarrow X$ is a closed subscheme then it is natural say that Z is non-degenerate if $Z_{\overline{k}}$ is reduced, i.e., if $Z \to k$ is geometrically reduced. If Z is of dimension zero then Z is non-degenerate if and only if $Z \to k$ is étale. Similarly for any scheme S, a finite flat morphism $Z \to S$ of finite presentation is called a non-degenerate family if every fiber is non-degenerate, or equivalently, if $Z \to S$ is étale.

Let $Z \to S$ be a family of zero dimensional subschemes, i.e., a finite flat morphism of finite presentation. The subset of S consisting of points $s \in S$ such that the fiber $Z_s \to k(s)$ is non-degenerate is open [$\mathrm{EGA}_{\mathrm{IV}}$, Thm. 12.2.1 (viii)]. Thus, there is an open subset $\mathrm{Hilb}^d(X/S)_{\mathrm{nd}}$ of $\mathrm{Hilb}^d(X/S)$ parameterizing non-degenerate families.

(5.2.2) *Non-degenerate families of cycles* — A zero-cycle $\mathscr{Z} = \sum_i a_i[z_i]$ on a k-scheme X is called non-degenerate if every point in the support of $\mathscr{Z}_{\overline{k}}$ has multiplicity one. Equivalently the multiplicities a_i are all one and the field extensions $k(z_i)/k$ are separable. It is clear that there is a one-to-one correspondence between non-degenerate zero-cycles on X and non-degenerated zero-dimensional subschemes of X.

Given a family of cycles $\mathscr{Z} \to S$, i.e., a morphism $S \to \mathrm{Sym}^d(X/S)$, $S \to \Gamma^d(X/S)$ or $S \to \mathrm{Chow}_{0,d}\bigl(X \hookrightarrow \mathbb{P}(\mathcal{E})\bigr)$, we say that it is non-degenerate family if $\mathscr{Z}_s$ is non-degenerate for every $s \in S$.

(5.2.3) *Degeneracy locus of cycles* — Let $X \to S$ be a morphism of schemes and let $\Delta \hookrightarrow (X/S)^d$ be the big diagonal, i.e., the union of all diagonals $\Delta_{ij} : (X/S)^{d-1} \to (X/S)^d$. It is clear that the image of Δ by $(X/S)^d \to \mathrm{Sym}^d(X/S)$ parameterizes degenerate cycles and that the open complement parameterizes non-degenerate cycles. We let $\mathrm{Sym}^d(X/S)_{\mathrm{nd}}$, $\Gamma^d(X/S)_{\mathrm{nd}}$ and $\mathrm{Chow}_{0,d}\bigl(X \hookrightarrow \mathbb{P}(\mathcal{E})\bigr)_{\mathrm{nd}}$ be the open subschemes of $\mathrm{Sym}^d(X/S)$, $\Gamma^d(X/S)$ and $\mathrm{Chow}_{0,d}\bigl(X \hookrightarrow \mathbb{P}(\mathcal{E})\bigr)$ respectively, parameterizing non-degenerate cycles.

We will now give an explicit cover of the degeneracy locus of $\mathrm{Sym}^d(X/S)$, $\Gamma^d(X/S)$ and $\mathrm{Chow}_{0,d}\bigl(X \hookrightarrow \mathbb{P}(\mathcal{E})\bigr)$. Some of the notation is inspired by [ES04, 2.4 and 4.1] and [RS07].

Definition (5.2.4). Let A be a ring and B an A-algebra. Let $\mathbf{x} = (x_1, x_2, \dots, x_d) \in B^d$. We define the symmetrization and anti-symmetrization operators from B^d to

$\mathrm{T}^d_A(B)$ as follows

$$s(\mathbf{x}) = \sum_{\sigma\in\mathfrak{S}_d} x_{\sigma(1)} \otimes_A x_{\sigma(2)} \otimes_A \cdots \otimes_A x_{\sigma(d)}$$
$$a(\mathbf{x}) = \sum_{\sigma\in\mathfrak{S}_d} (-1)^{|\sigma|} x_{\sigma(1)} \otimes_A x_{\sigma(2)} \otimes_A \cdots \otimes_A x_{\sigma(d)}.$$

As s and a are A-multilinear, s is symmetric and a is alternating it follows that we get induced homomorphisms, also denoted s and a

$$s \,:\, \mathrm{S}^d_A(B) \to \mathrm{TS}^d_A(B)$$
$$a \,:\, \textstyle\bigwedge^d_A(B) \to \mathrm{T}^d_A(B).$$

Remark **(5.2.5).** If d is invertible in A, then sometimes the symmetrization and anti-symmetrization operators are defined as $\frac{1}{d!}s$ and $\frac{1}{d!}a$. We will never use this convention. In [ES04] the tensor $a(\mathbf{x})$ is denoted by $\nu(\mathbf{x})$ and referred to as a norm vector.

Definition (5.2.6). Let A be a ring and B an A-algebra. Let $\mathbf{x} = (x_1, x_2, \dots, x_d) \in B^d$ and $\mathbf{y} = (y_1, y_2, \dots, y_d) \in B^d$. We define the following element in $\Gamma^d_A(B)$

$$\delta(\mathbf{x},\mathbf{y}) = \det\bigl(\gamma^1(x_i y_j) \times \gamma^{d-1}(1)\bigr)_{ij}.$$

Following [RS07] we call the ideal $I = I_A = \bigl(\delta(\mathbf{x},\mathbf{y})\bigr)_{\mathbf{x},\mathbf{y}\in B^d}$, the canonical ideal. As δ is multilinear and alternating in both arguments we extend the definition of δ to a function

$$\delta \,:\, \textstyle\bigwedge^d_A(B) \times \bigwedge^d_A(B) \to \mathrm{S}^2_A\bigl(\bigwedge^d_A(B)\bigr) \to \Gamma^d_A(B).$$

Proposition (5.2.7) ([ES04, Prop. 4.4])**.** *Let A be a ring, B an A-algebra and $\mathbf{x},\mathbf{y} \in B^d$. The image of $\delta(\mathbf{x},\mathbf{y})$ by $\Gamma^d_A(B) \to \mathrm{TS}^d_A(B) \hookrightarrow \mathrm{T}^d_A(B)$ is $a(\mathbf{x})a(\mathbf{y})$. In particular, $a(\mathbf{x})a(\mathbf{y})$ is symmetric.*

Lemma (5.2.8) ([ES04, Lem. 2.5])**.** *Let A be a ring and let B and A' be A-algebras. Let $B' = B \otimes_A A'$. Denote by $I_A \subset \Gamma^d_A(B)$ and $I_{A'} \subset \Gamma^d_{A'}(B') = \Gamma^d_A(B) \otimes_A A'$ the canonical ideals corresponding to B and B'. Then $I_A A' = I_{A'}$.*

Lemma (5.2.9). *Let S be a scheme and X and S' be S-schemes. Let $X' = X \times_S S'$. Let $\varphi \,:\, \Gamma^d(X'/S') = \Gamma^d(X/S) \times_S S' \to \Gamma^d(X/S)$ be the projection morphism. The inverse image by φ of the degeneracy locus of $\Gamma^d(X/S)$ is the degeneracy locus of $\Gamma^d(X'/S')$.*

Proof. Obvious as we know that a geometric point $\mathrm{Spec}(k) \to \Gamma^d(X/S)$ corresponds to a zero-cycle of degree d on $X \times_S \mathrm{Spec}(k)$. □

Lemma (5.2.10). *Let k be a field and let B be a k-algebra generated as an algebra by the k-vector field $V \subseteq B$. Let k'/k be a field extension and let $x_1, x_2, \dots, x_d$ be d distinct k'-points of $\mathrm{Spec}(B \otimes_k k')$. If k has at least $\binom{d}{2}$ elements then there is an element $b \in V$ such that the values of b at $x_1, x_2, \dots, x_d$ are distinct.*

Proof. For a vector space $V_0 \subseteq V$ we let $B_0 \subseteq B$ be the sub-algebra generated by V_0. There is a finite dimensional vector space $V_0 \subseteq V$ such that the images of $x_1, x_2, \dots, x_d$ in $\operatorname{Spec}(B_0 \otimes_k k')$ are distinct. Replacing V and B with V_0 and B_0 we can thus assume that V is finite dimensional. It is further clear that we can assume that $B = \mathrm{S}(V)$. The points $x_1, x_2, \dots, x_d$ then corresponds to vectors of $V^\vee \otimes_k k'$ and we need to find a k-rational hyperplane which does not contain the $\binom{d}{2}$ difference vectors $x_i - x_j$. A similar counting argument as in the proof of Lemma (3.1.6) shows that if k has at least $\binom{d}{2}$ elements then this is possible. □

Proposition (5.2.11). *Let A be a ring and B an A-algebra. Let $V \subset B$ be an A-submodule such that B is generated by V as an algebra. Consider the following three ideals of $\Gamma^d_A(B)$*

(i) *The canonical ideal $I_1 = \big(\delta(\mathbf{x}, \mathbf{y})\big)_{\mathbf{x},\mathbf{y} \in B^d}$.*
(ii) *$I_2 = \big(\delta(\mathbf{x}, \mathbf{x})\big)_{\mathbf{x} \in B^d}$.*
(iii) *$I_3 = \big(\delta(\mathbf{x}, \mathbf{x})\big)_{\mathbf{x}=(1,b,b^2,\dots,b^{d-1}),\ b \in V}$.*

The closed subsets determined by I_1 and I_2 coincide with the degeneracy locus of $\Gamma^d\big(\operatorname{Spec}(B)/\operatorname{Spec}(A)\big) = \operatorname{Spec}\big(\Gamma^d_A(B)\big)$. If every residue field of A has at least $\binom{d}{2}$ elements then so does the closed subset determined by I_3.

Proof. The discussion in (5.2.3) shows that it is enough to prove that the image of the ideals I_k by the homomorphism $\Gamma^d_A(B) \to \mathrm{TS}^d_A(B) \hookrightarrow \mathrm{T}^d_A(B)$ set-theoretically defines the big diagonal of $\operatorname{Spec}\big(\mathrm{T}^d_A(B)\big)$. By Proposition (5.2.7) the image of $\delta(\mathbf{x}, \mathbf{y})$ is $a(\mathbf{x})a(\mathbf{y})$. Thus the radicals of the images of I_1 and I_2 equals the radical of $J = \big(a(\mathbf{x})\big)_{\mathbf{x} \in B^d}$. It is further easily seen that J is contained in the ideal of every diagonal of $\operatorname{Spec}\big(\mathrm{T}^d_A(B)\big)$. Equivalently, the closed subset corresponding to J contains the big diagonal.

By Lemmas (5.2.8) and (5.2.9) it is enough to show the first part of the proposition after any base change $A \to A'$ such that $\operatorname{Spec}(A') \to \operatorname{Spec}(A)$ is surjective. We can thus assume that every residue field of A has at least $\binom{d}{2}$ elements. Both parts of the proposition then follows if we show that the closed subset corresponding to the ideal

$$K = \big(a(1, b, b^2, \dots, b^{d-1})\big)_{b \in V} \subseteq \mathrm{T}^d_A(B)$$

is contained in the big diagonal. As the formation of the ideal K commutes with base changes $A \to A'$ which are either surjections or localizations we can replace A with one of its residue fields and assume that A is a field with at least $\binom{d}{2}$ elements.

Let $\operatorname{Spec}(k) : x \to \operatorname{Spec}\big(\mathrm{T}^d_A(B)\big)$ be a point corresponding to d distinct k-points $x_1, x_2, \dots, x_d$ of $\operatorname{Spec}(B \otimes_A k)$. Lemma (5.2.10) shows that there is an element $b \in V$ which takes d distinct values $a_1, a_2, \dots, a_d \in k$ on the d points. The value of

$a(1,b,b^2,\dots,b^{d-1})$ at x is then

$$\sum_{\sigma\in\mathfrak{S}_d}(-1)^{|\sigma|}a_1^{\sigma(1)-1}a_2^{\sigma(2)-1}\dots a_d^{\sigma(d)-1}=\det\begin{pmatrix}1 & a_1 & a_1^2 & \dots & a_1^{d-1}\\ 1 & a_2 & a_2^2 & \dots & a_2^{d-1}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & a_d & a_d^2 & \dots & a_d^{d-1}\end{pmatrix}=\prod_{j<i}(a_i-a_j)$$

which is non-zero. Thus x is not contained in the zero-set of K. This shows that zero-set of K is contained in the big diagonal and hence that zero-set defined by K is the big diagonal. □

5.3. **Non-degenerated symmetric tensors and divided powers.**

Lemma (5.3.1). *Let A be a ring, B an A-algebra and $x,y\in\bigwedge_A^d(B)$. Then $\Gamma_A^d(B)_{\delta(x,y)}\to\mathrm{TS}_A^d(B)_{\delta(x,y)}$ is an isomorphism.*

Proof. Denote the canonical homomorphism $\Gamma_A^d(B)\to\mathrm{TS}_A^d(B)$ with φ. Let $f\in\mathrm{TS}_A^d(B)$. As the anti-symmetrization operator $a:\mathrm{T}_A^d(B)\to\mathrm{T}_A^d(B)$ is a $\mathrm{TS}_A^d(B)$-module homomorphism we have that $fa(x)=a(fx)$. By Proposition (5.2.7)

$$f\varphi\big(\delta(x,y)\big)=fa(x)a(y)=a(fx)a(y)=\varphi\big(\delta(fx,y)\big)$$

which shows that φ is surjective after localizing in $\delta(x,y)$.

To show that $\varphi_{\delta(x,y)}$ is injective, it is enough to show that the composition

$$\Gamma_A^d(B)_{\delta(x,y)}\to\mathrm{TS}_A^d(B)_{\delta(x,y)}\hookrightarrow\mathrm{T}_A^d(B)_{\delta(x,y)}$$

is injective. Choose a surjection $F\twoheadrightarrow B$ with F a flat A-algebra and let I be the kernel of $F\twoheadrightarrow B$. Let J be the kernel of $\mathrm{T}_A^d(F)\twoheadrightarrow\mathrm{T}_A^d(B)$.

Let $f\in J^G$. As $f\in J$ we can write f as a sum $f_1+f_2+\dots+f_n$ such that for every i we have that $f_i=f_{i1}\otimes f_{i2}\otimes\dots\otimes f_{id}\in\mathrm{T}_A^d(F)$ with $f_{ij}\in I$ for some j. Choose liftings $x',y'\in\bigwedge_A^d(F)$ of $x,y\in\bigwedge_A^d(B)$. Identifying $\Gamma_A^d(F)$ and $\mathrm{TS}_A^d(F)$, we have that $f\delta(x',y')=\delta(fx',y')$. This is a sum of determinants with elements in $\Gamma_A^d(F)$ such that in every determinant there is a row in which every element is in the ideal $\gamma^1(I)\times\gamma^{d-1}(1)$. Thus $\delta(fx',y')$ is in the kernel of $\Gamma_A^d(F)\twoheadrightarrow\Gamma_A^d(B)$ by (1.1.8). The image of f in $\Gamma_A^d(B)$ is thus zero after multiplying with $\delta(x,y)$. Consequently φ is injective after localizing in $\delta(x,y)$. □

Theorem (5.3.2). *Let X/S be a separated algebraic space. Then $\mathrm{Sym}^d(X/S)_{\mathrm{nd}}\to\Gamma^d(X/S)_{\mathrm{nd}}$ is an isomorphism.*

Proof. We can assume that S and X are affine (3.2.5). The theorem then follows from Proposition (5.2.11) and Lemma (5.3.1). □

Definition (5.3.3). Let A be any ring and $B=A[x_1,x_2,\dots,x_r]$. We call the elements $f\in\Gamma_A^d(B)$ of degree one, see Definition (1.3.2), *multilinear* or *elementary multisymmetric functions*. These are elements of the form

$$\gamma^{d_1}(x_1)\times\gamma^{d_2}(x_2)\times\dots\times\gamma^{d_n}(x_n)\times\gamma^{d-d_1-\dots-d_n}(1).$$

We let $\Gamma^d_A(A[x_1,x_2,\dots,x_n])_{\mathrm{mult.lin.}}$ denote the subalgebra of $\Gamma^d_A(A[x_1,x_2,\dots,x_n])$ generated by multi-linear elements.

Remark **(5.3.4).** If the characteristic of A is zero or more generally if $d!$ is invertible in A, then $\Gamma^d_A(A[x_1,x_2,\dots,x_n])_{\mathrm{mult.lin.}} = \Gamma^d_A(A[x_1,x_2,\dots,x_n])$ by Theorem (1.3.4).

Lemma (5.3.5). *Let A be a ring and $B = A[x_1,x_2,\dots,x_n]$. Let $b \in B_1$ and let $\mathbf{x} = (1,b,b^2,\dots,b^{d-1})$. Then $\big(\Gamma^d_A(B)_{\mathrm{mult.lin.}}\big)_{\delta(\mathbf{x},\mathbf{x})} \hookrightarrow \Gamma^d_A(B)_{\delta(\mathbf{x},\mathbf{x})}$ is an isomorphism.*

Proof. Let $f \in \Gamma^d_A(B) = \mathrm{TS}^d_A(B)$. We will show that f is a sum of products of multilinear elements after multiplication by a power of $\delta(\mathbf{x},\mathbf{x})$. As $f\delta(\mathbf{x},\mathbf{x}) = \delta(f\mathbf{x},\mathbf{x})$ and the latter is a sum of products of elements of the type $\gamma^1(c)\times\gamma^{d-1}(1)$ we can assume that f is of this type. As $c \mapsto \gamma^1(c)\times\gamma^{d-1}(1)$ is linear we can further assume that $c = x^\alpha$ for some non-trivial monomial $x^\alpha \in B$. It will be useful to instead assume that $c = x^\alpha b^k$ with $|\alpha| \geq 1$ and $k \in \mathbb{N}$. We will now proceed on induction on $|\alpha|$.

Assume that $|\alpha| = 1$. If $k = 0$ then $f = \gamma^1(x^\alpha b^k)\times\gamma^{d-1}(1)$ is multilinear. We continue with induction on k to show that $f \in \Gamma^d_A(B)_{\mathrm{mult.lin.}}$. We have that

$$\begin{aligned} f = \gamma^1(x^\alpha b^k)\times\gamma^{d-1}(1) = \big(\gamma^1(x^\alpha b^{k-1})\times\gamma^{d-1}(1)\big)\big(\gamma^1(b)\times\gamma^{d-1}(1)\big)\\ - \gamma^1(x^\alpha b^{k-1})\times\gamma^1(b)\times\gamma^{d-2}(1)\end{aligned}$$

and by induction it is enough to show that the last term is in $\Gamma^d_A(B)_{\mathrm{mult.lin.}}$. Similar use of the relation

$$\begin{aligned} \gamma^1(x^\alpha b^{k-\ell})\times\gamma^\ell(b)\times\gamma^{d-\ell-1}(1) = \big(\gamma^1(x^\alpha b^{k-\ell-1})\times\gamma^{d-1}(1)\big)\big(\gamma^{\ell+1}(b)\times\gamma^{d-\ell-1}(1)\big)\\ - \gamma^1(x^\alpha b^{k-\ell-1})\times\gamma^{\ell+1}(b)\times\gamma^{d-\ell-2}(1)\end{aligned}$$

with $1 \leq l \leq d-2$ and $l \leq k-1$ shows that it is enough to consider either $\gamma^1(x^\alpha)\times\gamma^k(b)\times\gamma^{d-k-1}(1)$ if $k \leq d-1$ or $\gamma^1(x^\alpha b^{k-d+1})\times\gamma^{d-1}(b)$ if $k > d-1$. The first element of these is multilinear and the second is the product of the multilinear element $\gamma^d(b)$ and $\gamma^1(x^\alpha b^{k-d})\times\gamma^{d-1}(1)$ which by the induction on k is in $\Gamma^d_A(B)_{\mathrm{mult.lin.}}$.

If $|\alpha| > 1$ then $x^\alpha = x^{\alpha'}x^{\alpha''}$ for some α',α'' such that $|\alpha'|,|\alpha''| < |\alpha|$. We have that

$$\begin{aligned} f = \gamma^1(c)\times\gamma^{d-1}(1) = \big(\gamma^1(x^{\alpha'}b^k)\times\gamma^{d-1}(1)\big)\big(\gamma^1(x^{\alpha''})\times\gamma^{d-1}(1)\big)\\ - \gamma^1(x^{\alpha'}b^k)\times\gamma^1(x^{\alpha''})\times\gamma^{d-2}(1).\end{aligned}$$

By induction it is enough to show that the last term is a sum of products of multilinear elements, after suitable multiplication by $\delta(\mathbf{x},\mathbf{x})$. Let $g = \gamma^1(x^{\alpha'}b^k)\times\gamma^1(x^{\alpha''})\times\gamma^{d-2}(1)$. Then $g\delta(\mathbf{x},\mathbf{x}) = \delta(g\mathbf{x},\mathbf{x})$ which is a sum of products of elements of the kind $\gamma^1(x^{\alpha'}b^{t'})\times\gamma^{d-1}(1)$ and $\gamma^1(x^{\alpha''}b^{t''})\times\gamma^{d-1}(1)$. By induction on $|\alpha|$ these are in $\big(\Gamma^d_A(B)_{\mathrm{mult.lin.}}\big)_{\delta(\mathbf{x},\mathbf{x})}$. □

Theorem (5.3.6). *Let X/S be quasi-projective morphism of schemes and let $X \hookrightarrow \mathbb{P}_S(\mathcal{E})$ be an immersion for some quasi-coherent $\mathcal{O}_S$-module $\mathcal{E}$ of finite type. Then $\Gamma^d(X/S)_{\mathrm{nd}} \to \mathrm{Chow}_{0,d}\bigl(X \hookrightarrow \mathbb{P}(\mathcal{E})\bigr)_{\mathrm{nd}}$ is an isomorphism.*

Proof. As Γ commutes with arbitrary base change and Chow commutes with flat base change we may assume that S is affine and, using Lemma (1.2.3), that every residue field of S has at least $\binom{d}{2}$ elements. If $\mathcal{E}' \twoheadrightarrow \mathcal{E}$ is a surjection of $\mathcal{O}_S$-modules then $\mathrm{Chow}_{0,d}\bigl(X \hookrightarrow \mathbb{P}(\mathcal{E})\bigr) = \mathrm{Chow}_{0,d}\bigl(X \hookrightarrow \mathbb{P}(\mathcal{E}')\bigr)$ by Definition (3.3.8) and we may thus assume that $\mathcal{E}$ is free. Further as $\mathrm{Chow}_{0,d}\bigl(X \hookrightarrow \mathbb{P}(\mathcal{E})\bigr)$ is the schematic image of $\Gamma^d(X/S) \hookrightarrow \Gamma^d\bigl(\mathbb{P}(\mathcal{E})/S\bigr) \to \mathrm{Chow}_{0,d}\bigl(\mathbb{P}(\mathcal{E})\bigr)$ we may assume that $X = \mathbb{P}(\mathcal{E}) = \mathbb{P}^n$.

By Proposition (3.3.10) and the assumption on the residue fields of $S = \mathrm{Spec}(A)$, the scheme $\mathrm{Chow}_{0,d}\bigl(\mathbb{P}(\mathcal{E})\bigr)$ is covered by affine open subsets over which the morphism $\Gamma^d\bigl(\mathbb{P}(\mathcal{E})/S\bigr) \to \mathrm{Chow}_{0,d}\bigl(\mathbb{P}(\mathcal{E})\bigr)$ corresponds to the inclusion of rings

$$\Gamma^d_A(A[x_1, x_2, \dots, x_n])_{\mathrm{mult.lin.}} \hookrightarrow \Gamma^d_A(A[x_1, x_2, \dots, x_n]).$$

The theorem now follows from Proposition (5.2.11) and Lemma (5.3.5). $\square$

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