Almost Good Moduli Spaces

AIM, Jan 28, 2021

§ Good moduli spaces (definition, properties, theorems, wall-crossing)

§ Almost good mod. spaces ——— " ———

§ Relation to NRGIT

§ Good moduli spaces (gms)

Def: $\underset{\text{alg stack}}{\mathcal{X}} \xrightarrow{\pi} \underset{\text{alg space}}{X}$ is a good moduli space if

(0) π is qcqs.

(cohomolog. affine)

(1) $\pi_* : QCoh\,\mathcal{X} \to QCoh\,X$ is exact.

(2) $\mathcal{O}_X \to \pi_* \mathcal{O}_{\mathcal{X}}$ is an isomorphism.

Slightly stronger

coh dim 0 (1') $R^i\pi_{\mathcal{X}} = 0 \quad \forall i > 0.$

univ. c.d. 0 (1'') $R^i\pi'_{\mathcal{X}} = 0 \quad \forall i > 0$

$$\forall \quad \begin{tikzcd} \mathcal{X}' \arrow[r] \arrow[d, "\pi'"'] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[d, "\pi"] \\ X' \arrow[r] & X. \end{tikzcd}$$

(1) $\Leftrightarrow$ (1') $\Leftrightarrow$ (1'') if X qs and $\underbrace{\Delta_\pi \text{ affine}}_{}$

include in definition of gms ??

Ex 1: (affine GIT)

- G lin reductive $\circlearrowright$ Spec A

$$\begin{tikzcd} \mathcal{X} = [\mathrm{Spec}\,A/G] \arrow[d, "\pi"', "gms"] \\ X = \mathrm{Spec}\,A^G \end{tikzcd} \qquad \pi_* : \mathrm{Mod}^G A \to \mathrm{Mod}\,A^G \text{ exact}$$

$$M \longmapsto M^G$$

Ex 2: (projective GIT)

- G lin reductive $\circlearrowright Y$ projective
- ample line bundle $\mathcal{L}$ on Y w/ G-action

$\Longleftrightarrow$ $\mathcal{L} = \mathcal{O}(1)|_X$ $\quad X \hookrightarrow \mathbb{P}^n$, $G \circlearrowright X$, $\circlearrowleft$ linearly

$$\begin{tikzcd} Y^s \arrow[r, phantom, "\subset"] \arrow[d] \arrow[dr, phantom, "\square"] & Y^{Ms} \arrow[r, phantom, "\subset"] \arrow[d] \arrow[dr, phantom, "\square"] & Y^{ss} \arrow[r, phantom, "\subset"] \arrow[d] \arrow[dr, phantom, "\square"] & Y \arrow[d] \\ \mathcal{Y}^s \arrow[r, phantom, "\subset"] \arrow[d, "cms"'] \arrow[dr, phantom, "\square"] & \mathcal{Y}^{Ms} \arrow[r, phantom, "\subset"] \arrow[d, "\text{``cms''}"] \arrow[dr, phantom, "\square"] & \mathcal{Y}^{ss} \arrow[r, phantom, "\subset"] \arrow[d, "\pi"', "gms"] & \mathcal{Y} = [Y/G] \\ Y^s/\!\!/G \arrow[r, phantom, "\subset"] & Y^{Ms}/\!\!/G \arrow[r, phantom, "\subset"] & Y^{ss}/\!\!/G & \end{tikzcd}$$

gms ← char 0 (o/w adequate m. sp.)

$Y^{ss}/\!\!/G$ ← projective $= \mathrm{Proj}\left(\bigoplus_{d \geq 0} \Gamma(Y, \mathcal{L}^d)^G\right)$

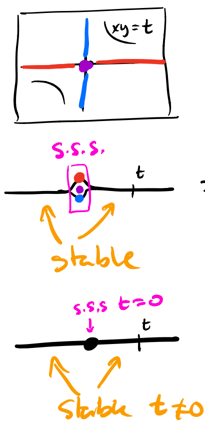
"cms" = gerbe under connected red group followed by cms

Main object $\mathcal{X} = \mathcal{Y}^{ss}$, $X = Y^{ss}/\!\!/G$

Def: ((Mumford) stable locus) $\mathcal{X} \xrightarrow{\pi} X$ gms.

- $X^{ms} \subset X$ open locus of $x \in X$: $\pi^{-1}(x) = \{x_0\}$
- $X^{s} \subset X$ —"— —"— + $\mathrm{stab}(x_0)$ finite

Ex: $\mathbb{G}_m \circlearrowright \mathbb{A}^2$ wts 1 and -1



$$
\begin{array}{c}
= \mathbb{A}^2 \\
\downarrow \\
= [\mathbb{A}^2/\mathbb{G}_m] = \mathcal{X} \\
\downarrow \pi \\
= \mathbb{A}^1 = X
\end{array}
$$

Prop (Alper '08) $\pi: \mathcal{X} \longrightarrow X$ gms

(universal) • π universal among maps to alg. spaces.

(topology)
- π universally closed
- $\forall x \in |X|$ $\exists!$ closed pt $x_0 \in \pi^{-1}(x)$ and $\mathrm{stab}(x_0)$ lin. red.

(finiteness)
- $\mathcal{X}$ noetherian $\Rightarrow$ X noetherian
- —"— $\Rightarrow$ π_* preserves coherence.
- $\mathcal{X} \longrightarrow \underset{\text{noeth}}{S}$ f.t. $\Rightarrow$ $X \longrightarrow S$ f.t.

(proj formula) • $(\pi_* F) \otimes G \xrightarrow{\cong} \pi_*(F \otimes \pi^* G)$ $\forall F \in QCoh(\mathcal{X})$, $G \in QCoh(X)$.

(base change)
- π' is a gms
- $\mathcal{X}' = \mathrm{Spec}_{\mathcal{X}} \mathcal{A}$ $\Rightarrow$ $X' = \mathrm{Spec}_X \pi_* \mathcal{A}$
- $g^* \pi_* \xrightarrow{\cong} \pi'_* g'^*$

$\forall$
$$
\begin{tikzcd}
\mathcal{X}' \arrow[r, "g'"] \arrow[d, "\pi'"'] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[d, "\pi"] \\
X' \arrow[r, "g"'] & X
\end{tikzcd}
$$

Thm (Local structure AHR'15) over a field $\mathcal{X} \longrightarrow X$ gms, $x \in |X|$

$G_x = \mathrm{stab}(x_0)$ (linearly reductive)

$$\exists \quad \begin{tikzcd} {[\mathrm{affine}/G_x]} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[d, "\pi"] \\ X' \arrow[r, "\text{étale}", "\text{nbhd of } x"'] & X \end{tikzcd}$$

(in general, replace G_x by lin. red. group scheme over X')

Thm (Partial desing. Kirwan '85, Edidin–R '17) $\mathcal{X} \longrightarrow X$ gms

$\exists$ canonical seq of saturated blow-ups (suppose $\mathcal{X}$ irr. for simp.)

$$\begin{tikzcd} \widetilde{\mathcal{X}} = \mathcal{X}_n \arrow[r] \arrow[d, "\pi_n"] & \cdots \arrow[r] & \mathcal{X}_1 \arrow[r] \arrow[d, "\pi_1"] & \mathcal{X}_0 = \mathcal{X} \arrow[d, "\pi"] \\ \widetilde{X} = X_n \arrow[r] & \cdots \arrow[r] & X_1 \arrow[r] & X_0 = X \end{tikzcd}$$

s.th. • $\mathcal{X}_{i+1} = Bl^{\pi_i}_{\mathcal{X}_i^{max}} \mathcal{X}_i \subset Bl_{\mathcal{X}_i^{max}} \mathcal{X}_i$

• max dim stab $\mathcal{X}_{i+1}$ < max dim stab $\mathcal{X}_i$

• if $X^s \neq \emptyset \Rightarrow \pi_n$ cms

• if $X^{Ms} \neq \emptyset \Rightarrow \pi_n$ "cms"

• in general $\widetilde{\mathcal{X}}^{max} \longrightarrow \widetilde{X}$ "cms"

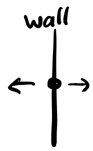
Cor: $\mathcal{X} \longrightarrow X$ gms, $\mathcal{X} \longrightarrow S$ smooth

$\Rightarrow \exists^{can} X' \longrightarrow X$ seq of blowups s.th. $X' \longrightarrow S$ smooth

Wall-crossing in GIT

$G \circlearrowright X$, $\mathcal{L}$ and $\chi : G \to \mathbb{G}_m$

$$X^{ss}_{\mathcal{L}^-} \subset X^{ss}_{\mathcal{L}} \supset X^{ss}_{\mathcal{L}^+} \qquad \mathcal{L}^{\pm} = \mathcal{L} \otimes \chi^{\pm\epsilon} \quad \left(= \mathcal{L}^{\otimes N} \otimes \chi^{\pm 1}\right)$$



Wall-crossing for stacks with gms

$\mathcal{X} = [X^{ss}_{\mathcal{L}}/G]$, χ gives line bundle on $\mathcal{X}$

$$\begin{array}{ccccc} \mathcal{X}^- & \subset & \mathcal{X} & \supset & \mathcal{X}^+ \\ \downarrow{\scriptstyle gms} & & \pi \downarrow{\scriptstyle gms} & & \downarrow{\scriptstyle gms} \\ X^- & \xrightarrow[proj]{} & X & \xleftarrow[proj]{} & X^+ \end{array}$$

$$\mathcal{X}^+ = \mathrm{Proj}^{\pi}_{\mathcal{X}}(\mathrm{Sym}\,\chi) \subset \underset{\substack{\parallel \\ \mathcal{X}}}{\mathrm{Proj}_{\mathcal{X}}}(\mathrm{Sym}\,\chi)$$

$$X^+ = \mathrm{Proj}_X\left(\bigoplus_{d \geq 0} \pi_*(\chi^d)\right)$$

More generally: $\mathcal{X} \xrightarrow{gms} X$, $\chi \in \mathrm{Pic}(\mathcal{X})$

§ Almost good moduli spaces

Def: $\pi : \mathcal{X} \longrightarrow X$ is

- almost coh affine if $\pi_* \mathcal{O}_{\mathcal{X}} \longrightarrow \pi_*(\mathcal{O}_{\mathcal{X}}/\mathcal{I})$ surj. $\forall \mathcal{I} \subset \mathcal{O}_{\mathcal{X}}$ + after b.c.
  $\Uparrow$
- almost coh dim 0 if $R^i\pi_*(\mathcal{I}) = 0$ $\forall i > 0$ — " —

Def: $\pi : \mathcal{X} \longrightarrow X$ (weak) almost good mod. space if ((w)agms)

(0) π qcqs

(weak) (1) π almost coh affine (1') π almost c.d. 0

(2) $\mathcal{O}_X \xrightarrow{\cong} \pi_* \mathcal{O}_{\mathcal{X}}$

Ex: gms $\Rightarrow$ agms.

Ex: $BG \longrightarrow *$ weak almost gms $\forall$ groups G.

Main new example: (char 0)

- H alg. group
  - $\mathcal{U}$ unipotent radical, $R = H/\mathcal{U}$ reductive, $H = \mathcal{U} \rtimes R$
- $\mathbb{G}_m \xhookrightarrow{\lambda} Z(R)$ central 1-ps s.th.
  $\lambda \circlearrowright \mathfrak{u}^\vee$ has strictly positive wts >0 (Kirwan: strictly neg. wts)
- V H-rep with λ-wts ≥ 0

$\Longrightarrow$ (1) $V^H = V^R$

(2) $H^i(H, V) = 0$ $\forall i \geq 1$

- $H \circlearrowright \operatorname{Spec} A$ with λ-wts ≥ 0

  $\Rightarrow H \circlearrowright I$ —〃— $\forall\, I \subseteq A$

  $\Rightarrow [\operatorname{Spec} A/H] \longrightarrow \operatorname{Spec} \underbrace{A^H}_{=A^R}$ agms

  In particular, $BH \longrightarrow *$ agms

- $H \circlearrowright Y$ projective, $H \circlearrowright \mathcal{L}$ ample with λ-wts ≥ 0

  $\Rightarrow [\underbrace{Y^{ss,H}}_{=Y^{ss,R}}/H] \longrightarrow \operatorname{Proj}\Big(\bigoplus_{d\geq 0} \underbrace{\Gamma(Y,\mathcal{L}^d)^H}_{=\Gamma(Y,\mathcal{L}^d)^R}\Big)$ agms

<u>Prop:</u> $\pi : \mathcal{X} \longrightarrow X$ (ω)agms

(universal) • π universal among maps to alg. spaces.

(topology)
- π universally closed
- $\forall x \in |X|$ $\exists!$ closed pt $x_0 \in \pi^{-1}(x)$ ~~and stab(x_0) lin. red~~.

(finiteness)
- $\mathcal{X}$ noetherian $\Rightarrow$ X noetherian
- —〃— $\Rightarrow$ π_* preserves coherence.
- $\mathcal{X} \longrightarrow \underset{\text{noeth}}{S}$ f.t. $\Rightarrow$ $X \longrightarrow S$ f.t.

(proj formula) • $(\pi_* F) \otimes G \xrightarrow{\cong} \pi_*(F \otimes \pi^* G)$ * $\forall F \in QCoh(\mathcal{X})$, $G \in QCoh(X)$.

(base change)
- π' is a (ω)agms
- $\mathcal{X}' = \operatorname{Spec}_{\mathcal{X}} \mathcal{A}$ $\Rightarrow$ $X' = \operatorname{Spec}_X \pi_* \mathcal{A}$ *
- $g^* \pi_* \xrightarrow{\cong} \pi'_* g'^*$ *

$\forall$
$$\begin{tikzcd} \mathcal{X}' \arrow[r, "g'"] \arrow[d, "\pi'"'] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[d, "\pi"] \\ X' \arrow[r, "g"'] & X \end{tikzcd}$$

* restricted to $QCoh^{\omega\text{-}\pi}(\mathcal{X})$

<u>Rmk</u>: $\pi_* \mathcal{A}$ f.g. if $\mathcal{A} \in QCoh^{\omega\text{-}\pi}(\mathcal{X})$

Def: ((weakly) π-good sheaves)

$$\mathrm{QCoh}^{w\text{-}\pi}(\mathcal{X}) = \{ F \in \mathrm{QCoh}\,\mathcal{X} : \pi_* F \to \pi_*(F/K) \text{ surj } \forall K \subset F \}$$

$$\mathrm{QCoh}^{\pi}(\mathcal{X}) = \{ F \in \mathrm{QCoh}\,\mathcal{X} : R^i\pi_* K = 0 \ \forall K \subset F \}$$

Lemma:
- Both closed under subsheaves, quotients, colimits.
- $\mathrm{QCoh}^{\pi}(\mathcal{X})$ also closed under extensions.

Warning: Not necc closed under $\otimes$, but closed under $\otimes \pi^*(-)$.

Rmk: π almost coh aff / almost c.d. 0 $\iff$ $\mathcal{O}_{\mathcal{X}}$ π-good / w. π-good

Ex (cont.) $H = U \rtimes R$, λ central 1-ps grading U
- Positive (wts ≥ 0) reps are π-good and closed under $\otimes$.
- Also $\exists$ non-positive π-good reps.

Lemma: $\mathcal{X} = BH$, $\underline{H = U \rtimes R}$ (no grading!), $V \in \mathrm{Rep}\, H$.

(1) V weakly π-good $\iff$ $W^H = W^R$ $\forall W \subseteq V$ subrep

(2) V π-good $\iff$ $H^i(H, W) = H^i(R, W)$ $\forall i \geq 0$ $\forall W \subseteq V$

Rmk: $BH \to *$ agms $\not\Rightarrow$ H as in example.

Q: $\exists$ "better" definition of agms also involving $\otimes$?

Q: If char $= p$, does agms $\Rightarrow$ gms ??

Ex: (saturated blow-ups)

$\mathcal{X} \longrightarrow X$ (w)agms, $\quad \mathcal{I} \subset \mathcal{O}_{\mathcal{X}}$ ideal

$\Rightarrow$ Rees algebra $\bigoplus_{n \geq 0} \mathcal{I}^n$ weakly π-good.

$$\Rightarrow \quad \begin{tikzcd} \mathrm{Bl}^{\pi}_{\mathcal{I}} \mathcal{O}_{\mathcal{X}} \arrow[r] \arrow[d, "\text{(w)agms}"'] \arrow[dr, phantom, "\circ"] & \mathcal{X} \arrow[d, "\text{(w)agms}"] \\ \mathrm{Bl}_{\pi_*(\mathcal{I}^N)} X \arrow[r] & X \end{tikzcd}$$

Ex: (wall-crossing)

$\mathcal{X} \to X$ (w)agms, $\chi \in \mathrm{Pic}\,\mathcal{X}$ s.th. χ^d weakly π-good $\forall d \geq 0$

$$\Rightarrow \quad \begin{tikzcd} \mathcal{X}_+ \arrow[r, phantom, "\overset{\text{open}}{\subset}"] \arrow[d, "\text{(w)agms}"'] & \mathcal{X} \arrow[d, "\pi"', "\text{(w)agms}"] \\ X_+ \arrow[r, "\text{proj}"'] & X \end{tikzcd}$$

$$X_+ = \mathrm{Proj}\big(\underbrace{\bigoplus \pi_* \chi^d}_{\text{f.g.}}\big)$$

§ Results for almost good moduli spaces

Thm (formal functions)

$$\begin{tikzcd} \mathcal{X}_0 \arrow[r, hook, "\mathcal{I}"] \arrow[d] \arrow[dr, phantom, "\circ"] & \mathcal{X} \arrow[d, "\pi"', "\text{wagms}"] \\ \mathrm{Spec}(A/I) = X_0 \arrow[r, hook, "I"] & X = \mathrm{Spec}\, A \end{tikzcd}$$

If A I-adically complete, then

$$\Gamma(\mathcal{X}, F) \xrightarrow{\cong} \varprojlim_n \Gamma(\mathcal{X}, F/\mathcal{I}^n F)$$

$\forall F \in QCoh^{w-\pi}(\mathcal{X})$

Thm (local structure) $\mathcal{X} \xrightarrow{\pi} X$ (w)agms, $x \in |X|$, Δ_π affine

$R = G_{x_0} / (\text{unipotent radical})$

$$\exists \quad \begin{tikzcd} {[\mathrm{affine}/R]} \arrow[d, "\text{smooth, affine, surj}"] \arrow[dd, bend right=60, "\text{gms}"'] & \\ \mathcal{X}' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[d, "\text{(w)agms}"] \\ X' \arrow[r, "\text{étale}", "\text{nbhd of } x"'] & X \end{tikzcd}$$

pf: (1) Apply (general) loc. str. to $BR \to BG_{x_0} \hookrightarrow \mathcal{X}$

$$\rightsquigarrow \quad \begin{tikzcd} \mathcal{W} = [\operatorname{Spec} A/R] \arrow[r, "h", "\text{smooth}"'] & \mathcal{X} \\ \{u\} = BR \arrow[u, hook] \arrow[r] & BG_{x_0} \arrow[u, hook] \end{tikzcd}$$

(2) Show that $\operatorname{Spec} A^R \to X$ is étale at image of u.

Use formal functions and:

$$BG_{x_0} = \mathcal{X}_0 \xhookrightarrow{m/m^2} \mathcal{X}_1 \xhookrightarrow{m^2/m^3} \mathcal{X}_2 \hookrightarrow \cdots \qquad \mathcal{X}$$

$$BR = \mathcal{W}_0 \xhookrightarrow{m'/m'^2} \mathcal{W}_1 \xhookrightarrow{m'^2/m'^3} \mathcal{W}_2 \hookrightarrow \cdots \qquad \mathcal{W} \qquad m' = h^*(m)$$

m'^i/m'^{i+1} as R-rep is restriction of m^i/m^{i+1} as G_{x_0}-rep

π warms $\Rightarrow$ m^i/m^{i+1} are weakly π-good G_{x_0}-reps

$\Rightarrow \pi_*(m^i/m^{i+1}) = (m'^i/m'^{i+1})^{G_{x_0}} = (m'^i/m'^{i+1})^R$

$\Rightarrow \operatorname{Spec} A^R \to X$ iso on tangent spaces

$\Rightarrow$ étale. $\square$

Rmk: Seems unlikely that $\mathcal{X}' = [\operatorname{Spec} A/G_{x_0}]$ in general

Conj: $\mathfrak{X} \xrightarrow{\pi} X$ wagms, Δ_π affine. Then

- π gms $\iff$ G_{x_0} linearly reductive $\forall x \in |X|$

Follows from
- Local structure
- Luna fundamental lemma (Conj).

Conj: Reduction of stabilizers via sat. blow-ups.

Need:
- $\mathfrak{X}$ smooth $\Rightarrow \mathfrak{X}^{max}$ smooth
- max dim stab drops after sat. blow-up in $\mathfrak{X}^{max}$.

§ Non-reductive GIT

- Internally graded:

  $H = U \rtimes R$, λ central 1-ps grading U

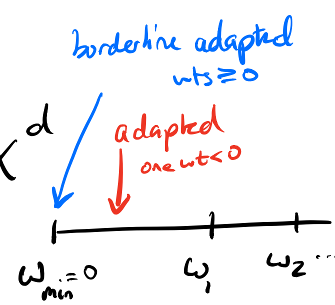
- Externally graded: choose grading $\mathbb{G}_m \curvearrowright U$.

  $\hat{H} = U \rtimes (R \times \mathbb{G}_m)$ $\quad \mathbb{G}_m$ grades U.

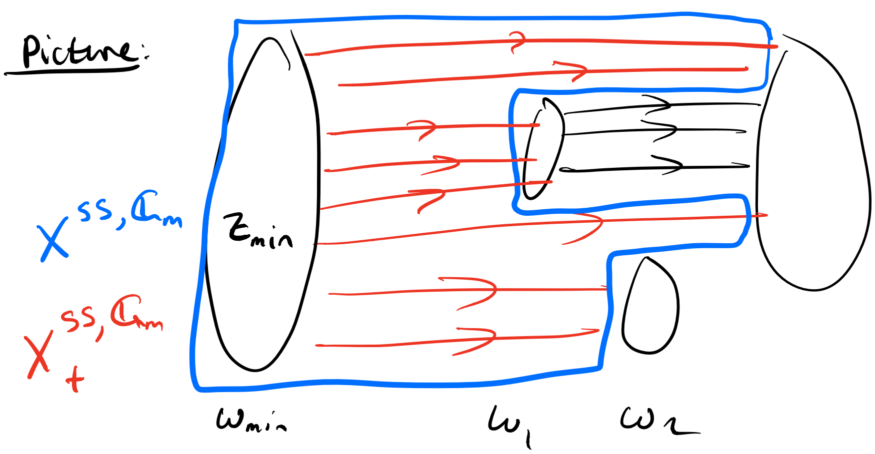
- χ character corresponding to λ.

- $H \curvearrowright Y$ projective, $H \curvearrowright \mathcal{L} \in \mathrm{Pic}(Y)$ ample.

- (Borderline adapted) Replace $\mathcal{L}$ with $\mathcal{L} \otimes \chi^d$ s.th. wts ≥ 0 including weight 0.



Picture:



$$
\begin{tikzcd}
\mathcal{X}' = [X^{ss,R}/R] \arrow[d] \arrow[dd, bend right, "\pi' \text{ gms}"'] & \supset \mathcal{X}'_+ & \\
\mathcal{X} = [X^{ss,R}/H] \arrow[d, "\pi"', "\text{agms}"] & \supset & \mathcal{X}_+ = \mathbb{P}^{\pi}(\mathcal{X}) \arrow[d, "\pi_+"] \\
X = X^{ss,R}/\!/R & & \mathrm{Proj}\Big(\underbrace{\bigoplus_{d\geq 0} \pi_* \chi^d}_{\subset \bigoplus_{d\geq 0} \pi'_* \chi^d}\Big) \arrow[ll]
\end{tikzcd}
$$

proj

Problem: χ is negative and in general not weakly π-good

$\not\Rightarrow$ $\bigoplus_{d\geq 0} \pi_* \chi^d$ f.g., π_+ agms.

$\Leftarrow$ closed pts of $\mathcal{X}$ have lin red stab.

Thm 1 (BDHK) If $\mathrm{stab}_{\mathcal{U}}(x) = \{e\}$ $\forall x \in X^{ss,R}$ then

- $\bigoplus_{d\geq 0} \pi_* \chi^d$ f.g. and π_+ gms.

Rmk: Conj $\Rightarrow$ π gms $\Rightarrow$ Thm 1

Thm 2 (BDHK) If $\dim \mathrm{stab}_{\mathcal{U}(j)}(x)$ constant $\forall x \in X^{ss,R}$ then

- $\bigoplus_{d\geq 0} \pi_* \chi^d$ f.g. and π_+ "gms"

unipotent gerbe + gms

Def: $X^{U\text{-stable}} = \{x \in |X| : \mathrm{stab}_U(x_0) = \{e\}\} \subset X$ (open)

($\underline{\text{Conj}}$: $\mathcal{X} \to X$ gms over U-stable)

Thm 3 (BDHK) If $X^{U\text{-stable}} \neq \emptyset$ then

$\exists$ sequence of saturated blow-ups of $\mathcal{X}$

$$\begin{tikzcd} \hat{\mathcal{X}} \arrow[r] \arrow[d, "\text{agms}"'] & \mathcal{X} \arrow[d, "\text{agms}"] \\ \hat{X} \arrow[r, "\text{blow-ups}"'] & X \end{tikzcd}$$

with centers $\{x : \dim \mathrm{stab}_U(x) \text{ maximal}\}$

(in part $\hat{X} \to X$ iso over $X^{U\text{-stable}}$)

s.th. $\hat{X}^{U\text{-stable}} = \hat{X}$, i.e. situation of Thm 1

$\Rightarrow$ $\hat{\pi}_+$ gms.

$$\begin{tikzcd} \hat{\mathcal{X}}_+ \arrow[d, "\text{gms}"'] & \mathcal{X}_+^{U\text{-sh.}} \arrow[l, hook'] \arrow[r, hook] \arrow[d, "\text{gms}"] & \mathcal{X}_+ \arrow[d, "\text{very bad quotient}"] \\ \hat{X}_+ & X_+^{U\text{-st}} \arrow[l, hook'] \arrow[r, hook] & X_+ \end{tikzcd}$$