FUNCTORIAL RESOLUTION OF SINGULARITIES IN CHARACTERISTIC ZERO USING REES ALGEBRAS

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ABSTRACT. This is a mixture of [Kol07, EV07, Kaw07, BM08] (and perhaps [Wło05]) used for the last two lectures for a mini-course on resolution of singularities. **DRAFT**

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1. Statement of main theorem

We work over any field k and consider schemes that are locally of finite type over the base field. We will occasionally need that k is of characteristic zero.

1.1. **Blow-up sequences.** Let X/k be a scheme, locally of finite type over the base field k. A *blow-up sequence* of length r on X is a sequence of blow-ups

$$\Pi\colon X_r \overset{\pi_{r-1}}{\to} X_{r-1} \overset{\pi_{r-2}}{\to} \dots \overset{\pi_1}{\to} X_1 \overset{\pi_0}{\to} X_0 = X$$

with *specified* centers $Z_i \hookrightarrow X_i$, so that $\pi_i\colon X_{i+1} = \mathrm{Bl}_{Z_i} X_i \to X_i$.

Remark **(1.1).** We allow a center Z_i to be a divisor or $Z_i = \emptyset$ but we consider two sequences to be equal if they only differ by blow-ups in empty centers.

Note that the composition Π does not determine the individual π_i nor does the π_i always determine the centers Z_i (a blow-up in a Cartier divisor is an isomorphism). Nevertheless, we will sometimes refer to the blow-up sequence as Π and the symbol π_i^* will depend on the center and not merely the morphism π_i. In particular, $\Pi^* = \pi_{r-1}^* \dots \pi_0^*$ depends on the full blow-up sequence and not merely the morphism Π.

We let $F_{i+1} = \pi_i^* Z_i$ be the exceptional divisor. The blow-up sequence is *smooth* if the centers are smooth.

For the definition of snc divisor and simple normal crossings, see §2.1. If $E \subseteq X$ is an snc divisor, then we let $E_0 = E$ and let $E_{i+1} = \pi_i^* E_i$ denote the total transform, namely $E_{i+1} = \mathrm{Bl}_{Z_i \cap E_i} E_i + F_{i+1}$ (here we ignore multiplicities). If E_i is snc and Z_i has simple normal crossings with E_i, then E_{i+1} is snc. We say that the blow-up sequence has simple normal crossings with E if every center Z_i has simple normal crossings with E_i.

If $E = \sum_j E^j$ is an ordered snc divisor, then we let $E_0 = \sum_j E^j$. Given $E_i = \sum_j E_i^j$, we define $E_{i+1} = \sum_j \mathrm{Bl}_{Z_i \cap E_i^j} E_i^j + F_{i+1}$ where F_{i+1} is given the highest ordering.

Let $\mathbb{B}$ be a blow-up sequence on X. We transform blow-up sequences in the following ways:

(i) Given a smooth morphism $p\colon X' \to X$ of k-schemes, we let $p^*\mathbb{B}$ denote the *pull-back* of $\mathbb{B}$, given by $X_i' = X_i \times_X X'$ and $Z_i' = Z_i \times_X X'$ (recall that blow-ups commute with flat base change). If $\mathbb{B}$ is smooth then so is $p^*\mathbb{B}$. If p is surjective, then $\mathbb{B}$ is uniquely determined by $p^*\mathbb{B}$ (by fppf descent).

(ii) Given a field extension k'/k, we let $\mathbb{B}_{k'}$ denote the pull-back of $\mathbb{B}$ along $X_{k'} \to X$, considered as a blow-up sequence of schemes over k'. Then $\mathbb{B}$ is uniquely determined by $\mathbb{B}_{k'}$ and if $\mathbb{B}$ is smooth, then so is $\mathbb{B}_{k'}$.

(iii) Given a closed subscheme $j\colon W \hookrightarrow X$, we let $j^*\mathbb{B}$ or $\mathbb{B}|_W$ denote the *restriction* of $\mathbb{B}$. This sequence has centers $Z_i \cap W_i$ and schemes $W_{i+1} = \mathrm{Bl}_{Z_i \cap W_i} W_i$. Note that W_{i+1} is the strict transform of $W_i \hookrightarrow X_i$ along the blow-up $X_{i+1} \to X_i$ so that $W_{i+1} \hookrightarrow X_{i+1}$.

(iv) Conversely, given a closed subscheme $j\colon W \hookrightarrow X$, and a blow-up sequence $\mathbb{B}_W$, we let $j_*\mathbb{B}_W$ denote the *push-forward* of $\mathbb{B}_W$ given by the same centers Z_i and $X_{i+1} = \mathrm{Bl}_{Z_i} X_i$. Note that $j^* j_* \mathbb{B}_W = \mathbb{B}_W$. If $\mathbb{B}_W$ is smooth, so is $j_* \mathbb{B}_W$.

Definition (1.2). A *blow-up sequence functor* is a functor $\mathcal{B}$ from triples $(X/k, \mathcal{I}, E)$ where X/k is a scheme, $\mathcal{I} \subseteq \mathcal{O}_X$ is a quasi-coherent ideal sheaf and E is an snc divisor on X, to blow-up sequences (of unspecified length) on X. We also consider the variant where the inputs are triples $(X, \mathcal{I}, \sum_{i=1}^n E_i)$ where $E = \sum_{i=1}^n E_i$ is an *ordered* snc divisor, i.e., E is snc and every E_i is smooth.

Definition (1.3). We say that $\mathcal{B}$ is functorial with respect to

(i) *smooth morphisms*, if $p^* \mathcal{B}(X, \mathcal{I}, E) = \mathcal{B}(X', p^* \mathcal{I}, p^* E)$ for every smooth morphism $p\colon X' \to X$ (recall that equality means up to empty blow-ups);
(ii) *change of fields*, if $\mathcal{B}(X/k, \mathcal{I}, E)_{k'} = \mathcal{B}(X_{k'}/k', p^* \mathcal{I}, p^* E)$ where $p\colon X_{k'} \to X$ is the induced morphism;
(iii) *closed embeddings*, if for any closed embedding $X \hookrightarrow Y$ of smooth k-schemes, any snc divisor $E \hookrightarrow Y$ such that $E|_X$ is snc and any closed subscheme $Z \hookrightarrow X$ with ideal sheaf $\mathcal{I}_X \subseteq \mathcal{O}_X$ and $\mathcal{I}_Y \subseteq \mathcal{O}_Y$, we have that $j_* \mathcal{B}(X, \mathcal{I}_X, E|_X) = \mathcal{B}(Y, \mathcal{I}_Y, E)$.

If $\mathcal{B}(X, \mathcal{I}_X, E|_X) = j^* \mathcal{B}(Y, \mathcal{I}_Y, E)$ holds in the last paragraph, then we say that $\mathcal{B}$ is weakly functorial with respect to closed embeddings. The difference is that some centers are allowed to lie outside Y.

1.2. **Main theorem.**

Theorem (1.4). *There is a blow-up sequence functor $\mathcal{BP}$ defined on all triples $(X, \mathcal{I}, E)$ where X is a smooth scheme of finite type over a field of characteristic zero and E is an unordered divisor with normal crossings. It satisfies the following conditions:*

(i) *The blow-up sequence $\Pi = \mathcal{BP}(X, \mathcal{I}, E)\colon X_r \to X$ has smooth centers and normal crossings with E.*
(ii) *The ideal $\mathcal{I}\mathcal{O}_{X_r}$ defines an snc divisor.*
(iii) Π *is an isomorphism over $X \setminus V(\mathcal{I})$.*
(iv) Π *is functorial with respect to smooth morphisms and change of fields.*
(v) Π *is weakly functorial with respect to closed embeddings.*

By a standard procedure, one deduces

Theorem (1.5). *There is a blow-up sequence functor $\mathcal{BR}$ defined on all schemes X of finite type over a field of characteristic zero. Let $\Pi = \mathcal{BR}(X)\colon X_r \to X$ denote the blow-up sequence. Then:*

(i) *X_r is smooth.*
(ii) $\Pi^{-1}(\mathrm{Sing}(X))$ *is an snc divisor.*
(iii) Π *is an isomorphism over $X \setminus \mathrm{Sing}(X)$.*
(iv) Π *is functorial with respect to smooth morphisms and change of fields.*

Note that the blow-up sequence $\mathcal{BR}(X)$ that we will construct need not have smooth centers [Kol07, Ex. 3.106]. It is possible to construct a smooth blow-up sequence functor $\mathcal{BR}'$ but then one has to be more careful and use a presentation of the Hilbert–Samuel function to control singularities [BM08, §1.3].

Proof of Theorem 1.5. First we construct $\mathcal{BR}(X)$ when $X = \operatorname{Spec}(A)$ is affine. We may then embed $X \hookrightarrow Y$ into a smooth affine k-scheme Y such that $\dim(Y) \geq \dim(X) + 2$. Consider the blow-up sequence $\mathcal{BP}(Y, \mathcal{I}, \emptyset)$ where $\mathcal{I}$ is the ideal sheaf defining X in Y. First assume that X is irreducible. As X is not a divisor, eventually the strict transform of X is going to be blown up. As the centers are smooth, this can only happen when either the strict transform of X is smooth or X is generically non-reduced. In the first case we stop the algorithm at this point and in the second case the strict transform of X becomes empty after performing the blow-up.

If X is not irreducible, the above will happen for every irreducible component W of X. If W is generically reduced, then we ignore the blow-up with center equal to the strict transform W' of W (more precisely, we replace the center Z with $Z \setminus W'$ since the center could be disconnected).

Using that $\mathcal{BP}$ is functorial with respect to closed embeddings, we may assume that $Y = \mathbb{A}^n_k$ for some n. To see that $\mathcal{BR}(X)$ is independent of the chosen embedding $X \hookrightarrow Y$, it is enough to show that two embeddings $X \hookrightarrow \mathbb{A}^{n_1}_k$ and $X \hookrightarrow \mathbb{A}^{n_2}_k$ give the same blow-up sequence. Using functoriality with respect to closed embeddings we may assume that $n = n_1 = n_2$ and after further increasing n, there is an automorphism of $\mathbb{A}^n_k$ that interchanges the two embeddings [Kol07, 3.39]. As $\mathcal{BP}$ is functorial with respect to smooth morphisms, it follows that the two blow-up sequences are equal.

That $\mathcal{BR}$ is functorial with change of fields follows from the corresponding fact for $\mathcal{BP}$. Given a smooth morphism $p\colon X' \to X$ and an embedding $X \hookrightarrow Y$ where Y is an affine smooth k-scheme, we can, locally on X', find a smooth affine morphism $Y' \to Y$ such that $X' = X \times_Y Y'$ [EGA$_{\mathrm{IV}}$, Prop. 18.1.1]. That $\mathcal{BR}$ is functorial with respect to smooth morphisms thus follows from the corresponding fact for $\mathcal{BP}$.

The extension of $\mathcal{BR}$ to arbitrary schemes is now a completely formal procedure using functoriality with respect to smooth morphisms (or merely open coverings) [Kol07, Prop. 3.37]. The same argument shows that $\mathcal{BR}$ extends to algebraic spaces and algebraic stacks. $\square$

1.3. **Outline of the proof of the main theorem.** There are two main ideas. The first is to use a very coarse invariant, the *order of vanishing* (Section 3). The order is an upper semi-continuous function $\operatorname{ord}_x(\mathcal{I})\colon X \to \mathbb{N}$ which is non-zero exactly over the support of $V(\mathcal{I})$. Let m be the maximum of the order function. The algorithm now proceeds with *order reduction*. It starts with the locus where $\operatorname{ord}_x(\mathcal{I}) = m$ and runs until the maximal order drops. It then continues with the locus of order $m-1$ and so on until the order equals one. This does not mean that $V(\mathcal{I})$ is snc, but it signifies that locally $V(\mathcal{I})$ is contained in a smooth hypersurface H and we may reduce $\mathcal{BP}(X, \mathcal{I}, E)$ to $\mathcal{BP}(H, \mathcal{I}|_H, E|_H)$ (after a "boundary modification", see below). Finally, we conclude by induction on the dimension of X.

To accomplish order reduction, the trick is to (locally) find a *hypersurface of maximal contact* H that contains the locus of maximal order and define the order reduction step as $\mathcal{BP}(H, \mathcal{I}|_H, E|_H)$. One uses derivatives to find the hypersurface of maximal contact and to show that it is essentially unique. Before we can pass to H, however, we need to make sure that $E|_H$ is an snc divisor. To accomplish this, we make a "boundary modification" that transforms $E + H$ into an snc divisor.

One then has to take care of the fact that the order of $\mathcal{I}|_H$ can be strictly larger than the order of $\mathcal{I}$ (the order depends on the embedding $V(\mathcal{I}) \hookrightarrow X$, not merely on $V(\mathcal{I})$). One way to do this is to use *marked ideals*. I have chosen to use *Rees algebras* as this gives, in my opinion, a more streamlined and elegant approach. The resulting algorithm is not quite identical to Kollár's as the boundary modification is done slightly differently. We also allow E to have self-intersections and the resulting algorithm is functorial with respect to all closed immersions even though E is not ordered. This answers [Kol07, 3.71].

The algorithm is described in greater detail in Section 6.

2. Differentials and coordinates for blow-ups

Let X/k be a smooth scheme.

2.1. **Regular system of parameters.** Assume that $X = \mathrm{Spec}(A)$. We say that $x_1, x_2, \dots, x_n \in A$ is a regular system of parameters if $dx_1, dx_2, \dots, dx_n$ is a basis for $\Omega^1_{X/S}$. If A is a regular local ring, then $x_1, x_2, \dots, x_n \in A$ is a regular system if and only if its image in $\mathfrak{m}_x/\mathfrak{m}_x^2 = \Omega^1_{X/S} \otimes \kappa(x)$ is a basis.

Locally on X (in the Zariski topology), there exists a regular system of parameters $x_1, x_2, \dots, x_n \in A$. Moreover, given a smooth closed subscheme $Z \hookrightarrow X$, we can locally always find a regular sequence such that $Z = V(x_1, \dots, x_r)$.

We say that a closed subscheme $E \hookrightarrow X$ is a divisor with *simple normal crossings* (snc) if locally on X there exists a regular system of parameters $x_1, x_2, \dots, x_n$ such that $E = V(x_1^{a_1} x_2^{a_2} \dots x_n^{a_n})$ for some $a_1, \dots, a_n \in \mathbb{N}$. Then $E = \sum_i a_i E_i$ where E_i are smooth divisors.

We say that a closed subscheme $Z \hookrightarrow X$ has *simple normal crossings* with an snc divisor E if there is a regular system as above where in addition $Z = V(x_1, x_2, \dots, x_r)$ for some $r \leq n$. In particular, Z is smooth and some of the E_i's are allowed to contain Z. If E does not contain Z, then $E|_Z$ is an snc divisor.

2.2. **Differential operators.** There is a general notion of differential operators $\mathcal{O}_X \to \mathcal{O}_X$ of order $\leq m$ and if X/k is smooth, then $\mathrm{Diff}^{\leq m}_{X/S}$ is locally free of finite rank. It is the dual of the sheaf of principal parts of length m. If $x_1, x_2, \dots, x_n \in A$ is a regular system of parameters, then $\mathrm{Diff}^{\leq m}_{X/S}$ is free with basis

$$\{\partial_{x^\alpha} \ : \ \alpha \in \mathbb{N}^n, |\alpha| \leq m\}$$

where (in characteristic zero)[1]

$$\partial_{x^\alpha} = \frac{1}{\alpha!} \cdot \frac{\partial^{|\alpha|}}{\partial x^\alpha} = \frac{1}{\alpha_1! \alpha_2! \dots \alpha_n!} \cdot \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_n^{\alpha_n}}$$

[1] In positive characteristic, the basis is described by $\partial_{x^\alpha}(x^\beta) = \binom{\beta}{\alpha} x^{\beta-\alpha}$.

If $\mathcal{I} \subseteq \mathcal{O}_X$ is a quasi-coherent ideal, then we let $D^m(\mathcal{I})$ be the ideal locally generated by $\partial(f)$ for all $\partial \in \mathrm{Diff}^{\leq m}$ and $f \in \mathcal{I}$. If $x_1, x_2, \dots, x_n$ is a regular system of parameters, and $\mathcal{I} = (f_1, f_2, \dots, f_s)$ then

$$D^m(\mathcal{I}) = (\partial_{x^\alpha}(f_j) \ : \ |\alpha| \leq m).$$

In particular, $\mathcal{I} = D^0(\mathcal{I}) \subseteq D^1(\mathcal{I}) \subseteq D^2(\mathcal{I}) \subseteq \dots$.

2.3. **Blow-ups.** Let $Z \hookrightarrow X$ be a smooth subscheme. Consider the blow-up $\pi\colon X' = \mathrm{Bl}_Z X \to X$ with the exceptional divisor $E = \pi^{-1}Z$.

Locally, $X = \mathrm{Spec}(A)$ and we have a regular system of parameters $x_1, x_2, \dots, x_n \in A$ such that $Z = V(x_1, x_2, \dots, x_r)$. Then X' is covered by charts $X'_i = D_+(x_i) = \mathrm{Spec}(A'_i) \subseteq X'$ where $i = 1, 2, \dots, r$ where $A'_i = A[x_j x_i^{-1} \ : \ j = 1, \dots, r] \subseteq A[x_i^{-1}]$. We let:

$$y_j = \begin{cases} x_j x_i^{-1}, & \text{if } 1 \leq j \leq r,\ j \neq i \\ x_j, & \text{if } j = i \text{ or } j > r. \end{cases}$$

Then $y_1, y_2, \dots, y_n$ is a regular system of parameters of X'_i. The exceptional divisor E is defined by $y_i = 0$ on X'_i.

With notation as above, let $f \in A$. Then $df = f_1 dx_1 + \dots f_n dx_n$ and on the chart $X'_1 = D_+(x_1)$ we have

$$\begin{aligned} \pi^*(f) &= f_1 dy_1 + \sum_{j=2}^{r} f_j d(y_1 y_j) + \sum_{j=r+1}^{n} f_j dy_j \\ &= (f_1 + f_2 y_2 + \dots + f_r y_r) dy_1 + \sum_{j=2}^{r} f_j y_1 dy_j + \sum_{j=r+1}^{n} f_j dy_j. \end{aligned}$$

Thus

$$\begin{aligned} \partial_{y_1}(\pi^* f) &= \pi^* \partial_{x_1}(f) + y_2 \pi^* \partial_{x_2}(f) + \dots + y_r \pi^* \partial_{x_r}(f) \\ \partial_{y_j}(\pi^* f) &= y_1 \pi^* \partial_{x_j}(f), \quad j = 2, \dots, r \\ \partial_{y_j}(\pi^* f) &= \pi^* \partial_{x_j}(f), \quad j = r+1, \dots, n \end{aligned}$$

or rearranged

$$\begin{aligned} \pi^* \partial_{x_1}(f) &= \partial_{y_1}(\pi^* f) - \frac{y_2}{y_1} \partial_{y_2}(\pi^* f) - \dots - \frac{y_r}{y_1} \partial_{y_r}(\pi^* f) \\ \pi^* \partial_{x_j}(f) &= \frac{1}{y_1} \partial_{y_j}(\pi^* f), \quad j = 2, \dots, r \\ \pi^* \partial_{x_j}(f) &= \partial_{y_j}(\pi^* f), \quad j = r+1, \dots, n. \end{aligned}$$

We also have the following version:

$$\begin{aligned} \pi^*(x_1 \partial_{x_1} f) &= y_1 \partial_{y_1}(\pi^* f) - y_2 \partial_{y_2}(\pi^* f) - \dots - y_r \partial_{y_r}(\pi^* f) \\ \pi^*(x_j \partial_{x_j} f) &= y_j \partial_{y_j}(\pi^* f), \quad j = 2, \dots, r \\ \pi^* \partial_{x_j}(f) &= \partial_{y_j}(\pi^* f), \quad j = r+1, \dots, n. \end{aligned} \tag{2.0.1}$$

which shows that logarithmic derivative behave well with respect to blow-ups. In fact, let $F = \sum F^j$ be an snc divisor on X and $Z = F^1 \cap F^2 \cap \dots \cap F^r$. Give X the log structure (X, F) and X' the log structure $(X', \widetilde{F} + E)$. Then $\pi^* \Omega^{\log}_{X/k} \to \Omega^{\log}_{X'/k}$ is an isomorphism which is reflected by the formulas above.

3. Order of vanishing and blow-ups

Let X/k be a smooth scheme, let $\mathcal{I} \subseteq \mathcal{O}_X$ be a quasi-coherent ideal sheaf and let $W = V(\mathcal{I}) \hookrightarrow X$ be the corresponding closed subscheme.

Definition (3.1). The *order of vanishing* of $\mathcal{I}$ at $x \in X$ is

$$\operatorname{ord}_x(W \hookrightarrow X) = \operatorname{ord}_x(\mathcal{I}) := \sup_{d \in \mathbb{N}} \{\mathcal{I}_x \subseteq \mathfrak{m}_x^d\}$$

We make the following easy observations.

- If $\mathcal{I} = (f_1, f_2, \dots, f_n)$ then $\operatorname{ord}_x(\mathcal{I}) = \min_i \operatorname{ord}_x(f_i)$.
- $\operatorname{ord}_x(\mathcal{I}) \geq 1$ if and only if $x \in \operatorname{Supp}(V(\mathcal{I}))$. Otherwise $\operatorname{ord}_x(\mathcal{I}) = 0$.
- $\operatorname{ord}_x(\mathcal{I}) = \infty$ if and only if $\mathcal{I}_x = 0$ (by Krull's intersection theorem).
- $\operatorname{ord}_x(W \hookrightarrow X)$ is not an invariant of W but depends on W and the embedding of W in X. This is in contrast with the tangent cone and multiplicity which only depend on W.
- $\operatorname{ord}_x(\mathcal{I}) = 1$ if and only if, locally (in the Zariski topology) around x, there is a smooth hypersurface $H \hookrightarrow X$ containing W. Indeed, any f such that $f_x \in \mathcal{I}_x \setminus \mathfrak{m}_x^2$ defines such a hypersurface.
- If $\mathcal{I}$ is locally principal (or equivalently, assume that W is a divisor), then $\operatorname{ord}_x(W \hookrightarrow X) = \operatorname{mult}_x(W)$ and $\operatorname{ord}_x(W \hookrightarrow X) = 1$ if and only if W is regular at x.

If $Z \hookrightarrow X$ is a smooth closed subscheme (possibly disconnected), then we say that $\operatorname{ord}_Z(W \hookrightarrow X) = m$ if $\operatorname{ord}_\xi(W \hookrightarrow X) = m$ for all generic points $\xi \in Z$.

The order of vanishing is tightly connected to differentials and blow-ups. First we note that, *in characteristic zero*, $\operatorname{ord}_x(D(\mathcal{I})) = \max\{\operatorname{ord}_x(\mathcal{I}) - 1, 0\}$. Thus, if $m \geq 1$, then the following are equivalent

(i) $\operatorname{ord}_x(\mathcal{I}) = m$,
(ii) $\operatorname{ord}_x(D(\mathcal{I})) = m - 1$,
(iii) $\operatorname{ord}_x(D^{m-1}(\mathcal{I})) = 1$.

and

$$\operatorname{ord}_x(\mathcal{I}) = \inf_{m \geq 0} \{D^m(\mathcal{I})_x = \mathcal{O}_{X,x}\}$$

The last formula is also valid in positive characteristic. In particular, it follows that $x \mapsto \operatorname{ord}_x(\mathcal{I})$ is an upper semi-continuous function.

Let $Z \hookrightarrow X$ be a smooth subscheme and let $\pi \colon X' = \operatorname{Bl}_Z X \to X$ be the blow-up in Z with exceptional divisor $E = \pi^{-1}Z$. Assume that $\operatorname{ord}_Z(W \hookrightarrow X) \geq m$. Then an easy calculation shows that $\pi^{-1}W$ contains mE. Indeed, for every $f \in \mathcal{I}$ we have that $\pi^* f = y_i^m \widetilde{f}$ in the chart $X'_i = D_+(x_i)$.

4. Marked ideals

Instead of taking strict transforms, we will keep track of the exceptional divisor in a more controlled way. In particular, we will use ideals where we pretend that their order is smaller than the actual order.

Definition (4.1). A *marked function* on X is a pair $(f, m) \in \Gamma(X, \mathcal{O}_X) \times \mathbb{N}$. A *marked ideal* on X is a pair $(\mathcal{I}, m)$ where $\mathcal{I} \subseteq \mathcal{O}_X$ and $m \in \mathbb{N}$. The *order* and *support* of a marked ideal are

$$\operatorname{ord}_x(\mathcal{I},m) = \frac{\operatorname{ord}_x(\mathcal{I})}{m}$$
$$\operatorname{Supp}(\mathcal{I},m) = \{x \in X \ : \ \operatorname{ord}_x(\mathcal{I}) \geq m\} = \{x \in X \ : \ \operatorname{ord}_x(\mathcal{I},m) \geq 1\}.$$

Note that the support is closed and that $\operatorname{Supp}(\mathcal{I},1) = \operatorname{Supp}(V(\mathcal{I}))$. We define

$$(\mathcal{I}_1,m) + (\mathcal{I}_2,m) = (\mathcal{I}_1 + \mathcal{I}_2, m)$$
$$(f_1,m_1) \cdot (f_2,m_2) = (f_1 f_2, m_1 + m_2)$$
$$(\mathcal{I}_1,m_1) \cdot (\mathcal{I}_2,m_2) = (\mathcal{I}_1\mathcal{I}_2, m_1 + m_2)$$

Note that

$$\operatorname{Supp}((\mathcal{I}_1,m) + (\mathcal{I}_2,m)) = \operatorname{Supp}(\mathcal{I}_1,m) \cap \operatorname{Supp}(\mathcal{I}_2,m)$$
$$\operatorname{Supp}((\mathcal{I}_1,m_1) \cdot (\mathcal{I}_2,m_2)) \supseteq \operatorname{Supp}(\mathcal{I}_1,m_1) \cap \operatorname{Supp}(\mathcal{I}_2,m_2)$$

but equality need not hold. This is fixed by introducing:

$$(\mathcal{I}_1,m_1) \odot (\mathcal{I}_2,m_2) = (\mathcal{I}_1^{m_2} + \mathcal{I}_2^{m_1}, m_1 + m_2)$$

for which $\operatorname{Supp}((\mathcal{I}_1,m_1) \odot (\mathcal{I}_2,m_2)) = \operatorname{Supp}(\mathcal{I}_1,m_1) \cap \operatorname{Supp}(\mathcal{I}_2,m_2)$. This notation is misleading as $\odot$ should be seen as the analogue of $\mathcal{I}_1+\mathcal{I}_2$. We will see a more elegant solution to these operations using *Rees algebras*. Marked ideals are also called Hironaka pairs.

Let X/k be a smooth scheme, let $Z \hookrightarrow X$ be a smooth closed subscheme and let $\pi\colon X' = \operatorname{Bl}_Z X \to X$ be the blow-up. Let $(\mathcal{I},m)$ be a marked ideal on X such that $Z \subseteq \operatorname{Supp}(\mathcal{I})$. Then $\pi^{-1}(V(\mathcal{I}))$ contains mE so that we can define the *controlled transform*

$$\pi^*(\mathcal{I},m) = (\pi^{-1}(\mathcal{I}) \otimes_{\mathcal{O}_{X'}} \mathcal{O}(mE), m)$$

that subtracts mE from $\pi^{-1}(V(\mathcal{I}))$. In local coordinates on the chart $X'_i = D_+(x_i)$ this means that $\pi^*(f,m) = (y_i^{-m}\pi^*(f), m)$.

4.1. **Derivatives of marked ideals.** Derivatives of marked ideals are defined as

$$D(\mathcal{I},m) = (D(\mathcal{I}), m-1), \quad D^j(\mathcal{I},m) = (D^j(\mathcal{I}), m-j)$$

so that $\operatorname{Supp}((\mathcal{I},m)) = \operatorname{Supp}(D(\mathcal{I},m)) = \dots = \operatorname{Supp}(D^{m-1}(\mathcal{I},m))$.

Translating the formulas from the last section, we now get more well-behaved derivatives with respect to blow-ups:

$$\begin{aligned}
y_1^{m-1}\pi^*(\partial_{x_1} f, m-1) &= \partial_{y_1}(y_1^m \pi^*(f,m)) - y_1^{m-1} y_2 \partial_{y_2} \pi^*(f,m) - \dots \\
&\qquad - y_1^{m-1} y_r \partial_{y_r} \pi^*(f,m) \\
y_1^{m-1}\pi^*(\partial_{x_j} f, m-1) &= y_1^{m-1}\partial_{y_j}\pi^*(f,m), \quad j = 2,\dots,r \\
y_1^{m-1}\pi^*(\partial_{x_j} f, m-1) &= y_1^{m}\partial_{y_j}\pi^*(f,m), \quad j = r+1,\dots,n
\end{aligned}$$

or equivalently:

$$\begin{aligned}
\pi^*(\partial_{x_1} f, m-1) &= (m,-1)\pi^*(f,m) + y_1\partial_{y_1}\pi^*(f,m) \\
&\qquad - y_2\partial_{y_2}\pi^*(f,m) - \dots - y_r\partial_{y_r}\pi^*(f,m) \\
\pi^*(\partial_{x_j} f, m-1) &= \partial_{y_j}\pi^*(f,m), \quad j = 2,\dots,r \\
\pi^*(\partial_{x_j} f, m-1) &= y_1\partial_{y_j}\pi^*(f,m), \quad j = r+1,\dots,n.
\end{aligned} \tag{4.1.1}$$

In particular, we have the important relation

$$\pi^*(\mathcal{I}, m) \subseteq \pi^* D(\mathcal{I}, m) \subseteq D(\pi^*(\mathcal{I}, m)).$$

4.2. **Logarithmic derivatives.** Let $H \hookrightarrow X$ be a smooth hypersurfaces. Then there is a notion of logarithmic derivations along H. If $x_1, x_2, \ldots, x_n$ is a regular system of parameters such that $H = V(x_1)$, then:

$$\mathrm{Der}_X(-\log H) = \big(x_1\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_n}\big)$$

and we define $D^n(-\log H)(\mathcal{I})$ as before using $\mathrm{Der}_X(-\log H)$. If $\mathcal{I}$ is generated by $f_1, \ldots, f_s$ then $D^m(-\log H)(f_i)$ is generated by $x_1^{\alpha_1}\partial_{x_\alpha} f_i$ for $|\alpha| \leq m$.

If $Z \hookrightarrow X$ is a smooth closed subscheme contained in H and $\pi\colon X' = \mathrm{Bl}_Z X \to X$ is the blow-up in Z, then the strict transform H' is a smooth hypersurface. If $x_1, x_2, \ldots, x_n$ is a regular system of parameters such that $Z = V(x_1, x_2, \ldots, x_r)$ and $H = V(x_r)$, then on the chart $D_+(x_i)$, $i = 1, 2, \ldots, r-1$ we have that $H' = V(y_r)$. On the chart $D_+(x_r)$ we have that $H' = \emptyset$. From the equations (4.1.1) it is immediately clear that we obtain the relation:

$$\pi^*(\mathcal{I}, m) \subseteq \pi^* D(-\log H)(\mathcal{I}, m) \subseteq D(-\log H)(\pi^*(\mathcal{I}, m))$$

(this also works if we replace H with an snc divisor E containing Z.)

The usefulness of log derivatives comes from the fact that they commute with restrictions along H

$$D(-\log H)(\mathcal{I})|_H = D(\mathcal{I}|_H)$$

since in local coordinates $x_1\partial_{x_1}(f)|_H = 0$.

5. Rees Algebras and Rees triples

A Rees algebra generalizes marked ideals by allowing sums of pairs. As before we assume that X is smooth over a field k (although many results hold without this).

Definition (5.1). A *Rees algebra* on X is a finitely generated graded $\mathcal{O}_X$-algebra

$$\mathcal{I}_\bullet = \bigoplus_{m \geq 0} \mathcal{I}_m t^m \subseteq \mathcal{O}_X[t]$$

such that $\mathcal{I}_0 = \mathcal{O}_X$. Furthermore, we let

$$\mathrm{ord}_x \mathcal{I}_\bullet = \inf_{m \geq 1} \mathrm{ord}_x(\mathcal{I}_m, m) = \inf_{m \geq 1} \frac{\mathrm{ord}_x(\mathcal{I}_m)}{m} = \inf_{\substack{f \in \mathcal{I}_m \\ m \geq 1}} \frac{\mathrm{ord}_x(f)}{m}$$

and

$$\begin{aligned}\mathrm{Supp}(\mathcal{I}_\bullet) &= \{x \in X \ : \ \mathrm{ord}_x \mathcal{I}_\bullet \geq 1\} \\ &= \{x \in X \ : \ \forall m,\ \mathrm{ord}_x \mathcal{I}_m \geq m\} = \cap_{m \geq 1} \mathrm{Supp}(\mathcal{I}_m, m)\end{aligned}$$

We will frequently denote an element $ft^m \in \mathcal{I}_m$ as (f,m) and we identify $(\mathcal{I},m)$ with the Rees algebra generated by $\mathcal{I}t^m$. Let $\mathcal{I}_\bullet$ be generated by $(f_1,m_1),\dots,(f_s,m_s)$. Then

$$\begin{aligned}
\operatorname{ord}_x \mathcal{I}_\bullet &= \min\{\operatorname{ord}_x(f_1,m_1),\dots,\operatorname{ord}_x(f_s,m_s)\} \\
&= \min\left\{\frac{\operatorname{ord}_x(f_1)}{m_1},\dots,\frac{\operatorname{ord}_x(f_s)}{m_s}\right\} \\
\operatorname{Supp}(\mathcal{I}_\bullet) &= \operatorname{Supp}(f_1,m_1)\cap\dots\operatorname{Supp}(f_s,m_s) \\
&= \{x\in X \ : \ \operatorname{ord}_x(f_1)\geq m_1,\dots,\operatorname{ord}_x(f_s)\geq m_s\}.
\end{aligned}$$

In particular, $\operatorname{ord}_x\mathcal{I}_\bullet \in \mathbb{Q}$ and $\operatorname{Supp}(\mathcal{I}_\bullet)$ is closed. We also let $\operatorname{maxord}(\mathcal{I}_\bullet) = \max_{x\in X}\operatorname{ord}_x\mathcal{I}_\bullet$. Note that we may now talk about the Rees algebra $(\mathcal{I}_1,m_1)+(\mathcal{I}_2,m_2)$ and that

$$\begin{aligned}
\operatorname{ord}_x((\mathcal{I}_1,m_1)+(\mathcal{I}_2,m_2)) &= \min\{\operatorname{ord}_x(\mathcal{I}_1,m_1),\operatorname{ord}_x(\mathcal{I}_2,m_2)\} \\
\operatorname{Supp}((\mathcal{I}_1,m_1)+(\mathcal{I}_2,m_2)) &= \operatorname{Supp}(\mathcal{I}_1,m_1)\cap\operatorname{Supp}(\mathcal{I}_2,m_2)
\end{aligned}$$

and similarly for sums of arbitrary Rees algebras.

If $\mathcal{I}$ is a usual ideal, then an element $f\in\mathcal{I}$ imposes the condition that $V(\mathcal{I})\subseteq V(f)$, i.e., $W=V(\mathcal{I})$ is contained in the vanishing locus of f. If $\mathcal{I}_\bullet$ is a Rees algebra, then similarly $(f,m)\in\mathcal{I}_\bullet$ signifies that $\operatorname{Supp}(\mathcal{I}_\bullet)\subseteq V(f,m)=\{x\in X \ :\operatorname{ord}_x(f)\geq m\}$, i.e., $\operatorname{Supp}(\mathcal{I}_\bullet)$ is contained in the locus where f vanishes to at least order m.

5.1. **Blow-ups.** Controlled transforms of marked ideals immediately extend to Rees algebras: for any blow-up $\pi\colon X'=\operatorname{Bl}_Z X\to X$ with smooth center Z contained in $\operatorname{Supp}(\mathcal{I}_\bullet)$ we define

$$\pi^*\mathcal{I}_\bullet = \bigoplus_{m\geq 0}\pi^*(\mathcal{I}_m,m)t^m.$$

5.2. **Rees triples.** A *Rees triple* R on a smooth scheme X is a triple $(\mathcal{I}_\bullet,E,F=\sum_j F^j)$ where

- $\mathcal{I}_\bullet$ is a Rees algebra,
- E is an unordered *normal crossings* divisor on X, and
- F is an ordered snc divisor on X,

such that $E+F$ is a normal crossings divisor.

A center $Z\hookrightarrow X$ is R-admissible, if:

- Z is smooth,
- $Z\subseteq\operatorname{Supp}(\mathcal{I}_\bullet)$, and
- Z has normal crossings with $E+F$.

The *transform* of R along $\pi\colon X'=\operatorname{Bl}_Z X\to X$ is $\pi^*R=(\pi^*\mathcal{I}_\bullet,\widetilde{E},\sum_j\widetilde{F^j}+F$ where $F=\pi^{-1}Z$ is the exceptional divisor and $\widetilde{E}=\overline{\pi^{-1}(E\setminus Z)}$ is the strict transform. Note that when $Z=F^j$, then $\widetilde{F^j}=\emptyset$ and $F=F^j$.

Let $\Pi=\pi_0\circ\dots\circ\pi_{r-1}\colon X_r\to X$ be a blow-up sequence. We say that the sequence is R-admissible if for every $0\leq i<r$, the center Z_i is $R_i:=\pi_{i-1}^*\dots\pi_1^*\pi_0^*R$-admissible. We let $\Pi^*R=R_r$.

5.3. **Projection from a marked line.** Let $(\mathbb{A}^1, 0)$ be $\mathbb{A}^1$ with a marked point. We let $X \times (\mathbb{A}^1, 0) = (X \times \mathbb{A}^1, X \times \{0\})$. These are smooth schemes together with a smooth divisor. Consider the projection $p\colon X \times (\mathbb{A}^1, 0) \to X$. Given a Rees triple $R = (\mathcal{I}_\bullet, E, F)$, we define $p^*R = (f^{-1}\mathcal{I}_\bullet, f^{-1}E, f^{-1}F + D)$ where $D = X \times \{0\}$.

5.4. **Exceptional blow-ups.** An exceptional blow-up [BM08, Def. 2.5] for a Rees triple $R = (\mathcal{I}_\bullet, E, F)$ is a blow-up in a center $Z = F^i \cap F^j$ which is the intersection of two exceptional divisors. Note that we do not require that $Z \subseteq \mathrm{Supp}(\mathcal{I}_\bullet)$. The transform of R along $\sigma\colon X' = \mathrm{Bl}_Z X \to X$ is $R' = (\sigma^*\mathcal{I}_\bullet, \widetilde{E}, \widetilde{F} + F')$ where $\sigma^*\mathcal{I}_\bullet = \bigoplus_{m \geq 0} \mathcal{I}_m \mathcal{O}_{X'} t^m$ is the *total transform*.

Remark **(5.2).** Smooth morphisms, projection from a marked line and exceptional blow-ups all induce log smooth morphisms $(X', F') \to (X, F)$.

5.5. **Equivalence of Rees triples.**

Definition (5.3). A *test sequence* for R is a sequence of transformations $\Pi = \pi_0 \circ \cdots \circ \pi_{r-1}$ such that π_i is either

- smooth,
- the projection from a marked line,
- an R_i-admissible blow-up, or
- an R_i-exceptional blow-up,

where $R_i := \pi_{i-1}^* \dots \pi_1^* \pi_0^* R$ denotes the transform (according to the type). A *weak test sequence* for R is a test sequence that only consists of smooth and admissible blow-ups.

Definition (5.4). We say that two Rees triples $R = (\mathcal{I}_\bullet, E, F)$ and $R' = (\mathcal{I}'_\bullet, E', F')$ are *equivalent*, written $R \sim R'$, if $F = F'$ (with the same ordering) and R and R' have the same test sequences. We say that $R \subseteq R'$ if $E = E'$, $F = F'$ and $\mathcal{I}_\bullet \subseteq \mathcal{I}'_\bullet$.

Lemma (5.5). *Let $R = (\mathcal{I}_\bullet, E, F) \sim R' = (\mathcal{I}'_\bullet, E', F')$ be two equivalent Rees triples.*

(i) $\mathrm{Supp}(R) = \mathrm{Supp}(R')$,
(ii) $\mathrm{ord}_x R = \mathrm{ord}_x R'$ *for all* $x \in \mathrm{Supp}(R)$,
(iii) $\mathrm{ord}_{F_j} R = \mathrm{ord}_{F_j} R'$ *for all exceptional divisors* F_j.
(iv) *If* $p\colon X' \to X$ *is smooth, then* $p^*R = p^*R'$.
(v) *If* $\pi\colon X' \to X$ *is an R-admissible blow-up, then* $\pi^*R \sim \pi^*R'$.
(vi) $(\mathcal{I}_\bullet, E + F, \emptyset) \sim (\mathcal{I}'_\bullet, E' + F', \emptyset)$.
(vii) *If* $R \subseteq R' \subseteq R''$, *and* $R \sim R''$, *then* $R \sim R'$.

Proof. (i) If $x \in \mathrm{Supp}(R)$ is a closed point, then the sequence $\mathrm{Bl}_x X \to X$ is admissible. Hence also $x \in \mathrm{Supp}(R')$. It follows that $\mathrm{Supp}(R) = \mathrm{Supp}(R')$.

(ii) and (iii): See appendix.

(iv), (v) and (vi) are obvious.

(vii): Trivially $\mathrm{Supp}(R) \supseteq \mathrm{Supp}(R') \supseteq \mathrm{Supp}(R'')$, but as $R \sim R''$, these are all equal. It follows that test sequences of length 1 for R and R' are equal. As the inclusions are preserved under transformations, it follows that $R \sim R'$. $\square$

If R and R' are only weakly equivalent, then everything except $\operatorname{ord}_{F_j} R = \operatorname{ord}_{F_j} R'$ holds.

5.6. **Integral closure of Rees algebras.**

Definition (5.6). Let $\mathcal{I}_\bullet$ be a Rees algebra. We let $\mathrm{IC}(\mathcal{I}_\bullet)$ be the *integral closure* of $\mathcal{I}_\bullet$ in $\mathcal{O}_X[t]$. We say that $\mathcal{I}_\bullet$ is integrally closed if $\mathcal{I}_\bullet = \mathrm{IC}(\mathcal{I}_\bullet)$.

Lemma (5.7). *The $\mathcal{O}_X$-algebra $\mathrm{IC}(\mathcal{I}_\bullet)$ is an integrally closed Rees algebra.*

Proof. It is well-known that the integral closure of a graded ring is graded [Bou64, p. 30] (high-tech reason: integral closure commutes with smooth morphisms and a $\mathbb{Z}$-grading is equivalent to a $\mathbb{G}_m$-action). The non-obvious fact is that $\mathrm{IC}(\mathcal{I}_\bullet)$ is finitely generated.

The question is local on X so we may assume that $X = \operatorname{Spec}(A)$ and that X is connected with fraction field K. Since X is smooth, X is normal so that $\mathcal{O}_X[t] \subseteq K(t)$ is integrally closed. Thus, $\mathrm{IC}(\mathcal{I}_\bullet)$ is the integral closure of $\mathcal{I}_\bullet$ in $K(t)$.

Next, we note that $\mathcal{I}_\bullet$ is an integral domain as $\mathcal{I}_\bullet \subseteq \mathcal{O}_X[t] \subseteq K[t]$. Let L denote the fraction field of $\mathrm{IC}(\mathcal{I}_\bullet)$. If $\mathcal{I}_\bullet = \mathcal{O}_X$ then $\mathrm{IC}(\mathcal{I}_\bullet) = \mathcal{I}_\bullet = \mathcal{O}_X$ and the result is clear. Otherwise, there exists $(f,m) \in \mathcal{I}_\bullet$ with $m \geq 1$. After inverting (f,m) and $(f,1)$ we have the function t^m so $K(t^m) \subseteq L \subseteq K(t)$.

Thus $K(t)/L$ is a finite field extension and since $\operatorname{Spec}_X(\mathcal{I}_\bullet)$ is of finite type over $\operatorname{Spec}(k)$, the integral closure of $\mathcal{I}_\bullet$ in $K(t)$ is finitely generated. $\square$

Definition (5.8). We say that two Rees algebras $\mathcal{I}_\bullet$ and $\mathcal{J}_\bullet$ are *integrally equivalent*, written $\mathcal{I}_\bullet \simeq \mathcal{J}_\bullet$, if $\mathrm{IC}(\mathcal{I}_\bullet) = \mathrm{IC}(\mathcal{J}_\bullet)$. Given a Rees triple $R = (\mathcal{I}_\bullet, E, F)$ we let $\mathrm{IC}(R) = (\mathrm{IC}(\mathcal{I}_\bullet), E, F)$ and write $(\mathcal{I}_\bullet, E, F) \simeq (\mathcal{I}'_\bullet, E', F')$ if $\mathcal{I}_\bullet \simeq \mathcal{I}'_\bullet$, $E = E'$ and $F = F'$.

Lemma (5.9). (i) *We have that $R \sim \mathrm{IC}(R)$.*
(ii) *If $\Pi\colon X_r \to X$ is a test sequence for R, then $\Pi^* R \subseteq \Pi^*(\mathrm{IC}(R)) \subseteq \mathrm{IC}(\Pi^* R)$*
(iii) *If $R \simeq R'$ then $R \sim R'$.*

Proof. We will prove (i) and (ii) simultaneously: we need (i) to even define $\Pi^*(\mathrm{IC}(R))$ in (ii). As integral closure commutes with smooth pull-back, it is enough to consider a single admissible or exceptional blow-up. Let $(f,m) \in \mathrm{IC}(\mathcal{I}_\bullet)$. Then there exists some positive integer d and a relation:

$$f^d + a_1 f^{d-1} + a_2 f^{d-2} + \cdots + a_d = 0$$

with $a_i \in \mathcal{I}_{mi}$. Trivially, $\operatorname{Supp}(\mathrm{IC}(\mathcal{I}_\bullet)) \subseteq \operatorname{Supp}(\mathcal{I}_\bullet)$. For the reverse inclusion, pick $x \in \operatorname{Supp}(\mathcal{I}_\bullet)$. Then $\operatorname{ord}_x(a_i) \geq mi$ and from the relation above it follows that $\operatorname{ord}_x(f) \geq m$. As this holds for any element in $\mathrm{IC}(\mathcal{I}_\bullet)$ it follows that $\operatorname{Supp}(\mathcal{I}_\bullet) = \operatorname{Supp}(\mathrm{IC}(\mathcal{I}_\bullet))$. Thus, a blow-up is admissible or exceptional for R if and only if it is so for $\mathrm{IC}(R)$.

After an R-admissible or R-exceptional blow-up $\pi\colon X' \to X$, the relation is transformed to

$$\pi^*(f,m)^d + \pi^*(a_1,m)\pi^*(f,m)^{d-1} + \pi^*(a_2,2m)\pi^*(f,m)^{d-2} + \cdots + \pi^*(a_d,dm) = 0$$

so $\pi^*(f) \in \mathrm{IC}(\pi^* \mathcal{I}_\bullet)$ and the inclusion in (ii) follows for test sequences of length 1. After an R-exceptional blow-up, It is thus enough to show that $\pi^*R \sim \mathrm{IC}(\pi^*R)$ and we conclude by induction on the length of a test sequence.

(iii) is a direct consequence of (i) since $R \simeq \mathrm{IC}(R) = \mathrm{IC}(R') \simeq R$. $\square$

Remark **(5.10).** Let $\mathcal{I}_\bullet^\natural = \bigoplus_m \big(\sum_k \mathcal{I}_{m+k}\big)t^m$. That is, if $\mathcal{I}_\bullet$ is generated by $(f_1, m_1), \dots, (f_n, m_n)$ then $\mathcal{I}_\bullet^\natural$ is generated by the elements (f_i, j) where $1 \leq j \leq m_i$. Clearly $\mathcal{I}_\bullet \sim \mathcal{I}_\bullet^\natural$ as the added elements do not impose any new conditions. Also it is easy to see that $\mathcal{I}_\bullet^\natural \subseteq \mathrm{IC}(\mathcal{I}_\bullet)$.

A more natural way is to introduce *extended Rees algebras*. These are finitely generated graded $\mathcal{O}_X$-algebras $\mathcal{I}_\bullet = \bigoplus \mathcal{I}_m t^m \subseteq \mathcal{O}_X[t, t^{-1}]$ such that $\mathcal{I}_m = \mathcal{O}_X$ for $m \leq 0$. In particular, $t^{-1} \in \mathcal{I}_\bullet$ so $\mathcal{I}_\bullet = \mathcal{I}_\bullet^\natural$. The idealistic filtrations of Kawanoue [Kaw07] (of r.f.g. type) are extended Rees algebras except that the $\mathbb{Z}$-grading is replaced with a $\mathbb{Q}$-grading.

5.7. **Derivatives.** When it comes to derivatives, it is convenient to add all derivatives at once:

$$D(\mathcal{I}_\bullet) = \bigoplus_{m \geq 0} \left(\sum_{k \geq 0} D^k(\mathcal{I}_{m+k}, m+k) \right) t^m$$

This is indeed a Rees algebra: if $\mathcal{I}_\bullet$ is generated by $(f_1, m_1), \dots, (f_s, m_s)$ and $x_1, x_2, \dots, x_n$ is a regular system of parameters, then $D(\mathcal{I}_\bullet)$ is generated by $(\partial_{x^\alpha} f_i, m_i - |\alpha|)$ for $|\alpha| < m_i$. If $\mathcal{I}_\bullet = D(\mathcal{I}_\bullet)$ then we say that $\mathcal{I}_\bullet$ is *differentially closed*. In particular, $D(\mathcal{I}_\bullet)$ is differentially closed. Similarly, given an snc (or even normal crossings) divisor $F \hookrightarrow X$, we define $D_F(\mathcal{I}_\bullet)$ and D_F-closed Rees algebras. An important case will be $F = H$ for a smooth (non-exceptional) hypersurface H. Note that $D = D_\emptyset$.

Given a Rees triple $R = (\mathcal{I}_\bullet, E, F)$, we let $D_F(R) = (D_F(\mathcal{I}_\bullet), E, F)$.

Lemma (5.11).
(i) *We have that $R \sim D_F(R)$.*
(ii) *If $\Pi \colon X' \to X$ is a test sequence for R then $\Pi^*R \subseteq \Pi^*(D_F(R)) \subseteq D_{F'}(\Pi^*R)$*
(iii) *If $R \sim R'$ then $D_F(R) \sim D_F(R')$.*

Proof. As with integral closure, we will prove (i) and (ii) simultaneously. We note that $\mathrm{Supp}(\mathcal{I}_\bullet) = D_F(\mathrm{Supp}(\mathcal{I}_\bullet))$ (§4.1). Thus, any R-admissible blow-up is $D_F(R)$-admissible. After one admissible blow-up the inclusion in (ii) holds by the corresponding result for marked ideals (§4.1). For exceptional blow-ups, it is an easy consequence of the transformation rule (2.0.1).

By induction on the length of a test sequence, it follows that $\Pi^*R \sim D_{F'}(\Pi^*R)$ and we conclude that $\Pi^*R \sim \Pi^*(D_F(R))$ so that $R \sim D_F(R)$.

(iii) is an immediate consequence of (i). $\square$

Remark **(5.12).** It can be shown that if $\mathcal{I}_\bullet \simeq \mathcal{J}_\bullet$ then $D(\mathcal{I}_\bullet) \simeq D(\mathcal{J}_\bullet)$. In fact, the following stronger result $D(\mathrm{IC}(\mathcal{I}_\bullet)) \subseteq \mathrm{IC}(D(\mathcal{I}_\bullet)) = D(\mathrm{IC}(D(\mathcal{I}_\bullet))) = \mathrm{IC}(D(\mathrm{IC}(\mathcal{I}_\bullet)))$ holds [Kaw07, Prop. 2.2.3.1, Cor. 2.2.3.2, Cor. 2.3.2.7].

We do not need this. In any case, if $\mathcal{I}_\bullet \simeq \mathcal{J}_\bullet$, then $\mathcal{I}_\bullet \sim \mathcal{J}_\bullet$ and so $D(\mathcal{I}_\bullet) \sim D(\mathcal{J}_\bullet)$ which is what is important for us.

5.8. **Simple Rees algebras.** A Rees algebra $\mathcal{I}_\bullet$ such that $\operatorname{maxord}(\mathcal{I}_\bullet) = 1$ is called *simple* by Encinas and Villamayor [EV07]. We leave the proof of the following easy lemma as an exercise.

Lemma (5.13). *Let $\mathcal{I}_\bullet$ be such that* $\operatorname{maxord}(\mathcal{I}_\bullet) \leq 1$. *Then* $\operatorname{ord}_x D(\mathcal{I}_\bullet)$ *is either* 0 *and* 1 *for every* $x \in X$.

This shows that D-closed simple Rees algebras behave very similarly to ordinary ideals.

5.9. **Fractions.** Given a Rees algebra $\mathcal{I}_\bullet$. Let $0 < m_1 < m_2 < \cdots < m_n \in \mathbb{N}$ be the degrees of a minimal set of generators of $\mathcal{I}_\bullet$, that is, m_i belongs to the sequence exactly when $(\mathcal{I}_n, n)$ is not in the subalgebra generated by $\sum_{m\geq 0}^{n-1} (\mathcal{I}_m, m)$. We let $\Delta(\mathcal{I}_\bullet) = \operatorname{lcm}(m_1, m_2, \dots, m_n)$. If Π is a test sequence for $R = (\mathcal{I}_\bullet, E, F)$, then clearly $\Delta(\Pi^*\mathcal{I}_\bullet) \mid \Delta(\mathcal{I}_\bullet)$. However, Δ is not preserved under equivalence. The importance of Δ is that $\operatorname{ord}_x(\mathcal{I}_\bullet) \in \frac{1}{\Delta}\mathbb{N}$.

Also, $\mathcal{I}_\bullet \simeq (\mathcal{I}_\Delta, \Delta)$ so this let us pass between Rees algebras and marked ideals. In most treatments, including [Kol07, BM08, Wło05], the marked ideal $(\mathcal{I}_\Delta, \Delta)$ is used instead of the Rees algebra $\mathcal{I}_\bullet$. Exceptions are [EV07, Kaw07] who use Rees algebras.

Given a Rees algebra $\mathcal{I}_\bullet$ and a positive rational number $\alpha \in \mathbb{Q}$ we let

$$\mathcal{I}_\bullet^\alpha = \bigoplus_{m\geq 0} (\mathcal{I}_{\lceil \alpha m \rceil}, m)$$

We do not necessarily have that $\mathcal{I}_\bullet^{\alpha\beta} = (\mathcal{I}_\bullet^\alpha)^\beta$ but at least $\mathcal{I}_\bullet^{\alpha\beta} \simeq (\mathcal{I}_\bullet^\alpha)^\beta$. Indeed, it is not difficult to see that $\mathcal{I}_\bullet^\alpha \simeq \mathcal{J}_\bullet^\alpha$ if $\mathcal{I}_\bullet \simeq \mathcal{J}_\bullet$ and

$$(\mathcal{I}_\Delta, \Delta)^{p/q} \simeq (\mathcal{I}_\Delta^p, q\Delta)$$

Note that $\operatorname{ord}_x(\mathcal{I}_\bullet^\alpha) = \alpha \operatorname{ord}_x \mathcal{I}_\bullet$.

5.10. **Monomial decompositions.** Let $R = (\mathcal{I}_\bullet, E, F = \sum_{j=1}^r F^j)$ be a Rees triple. Let $\beta_j = \operatorname{ord}_{F_j} \mathcal{I}_\bullet$. Recall that the β_j's are invariants of $\mathcal{I}_\bullet$ up to equivalence. The *monomial part* of R, denoted $M_F(\mathcal{I}_\bullet)$, is the Rees algebra $\prod_{j=1}^r (\mathcal{O}(-F^j), 1)^{\beta_j}$. Explicitly:

$$M_F(\mathcal{I}_\bullet)_m = \mathcal{O}(-\sum_j \lceil m\beta_j \rceil F^j)$$

The non-monomial part of R, denoted $N_F(\mathcal{I}_\bullet)$, is the Rees algebra $\mathcal{I}_\bullet / M_F(\mathcal{I}_\bullet)$. Explicitly:

$$N_F(\mathcal{I}_\bullet)_m = \mathcal{I}_m(\sum_j \lceil m\beta_j \rceil F^j)$$

If $\mathcal{I}_\bullet \sim \mathcal{J}_\bullet$, then $M_F(\mathcal{I}_\bullet) \sim M_F(\mathcal{J}_\bullet)$ and $N_F(\mathcal{I}_\bullet) \sim N_F(\mathcal{J}_\bullet)$.

For any sequence of test transformation $\Pi\colon X' \to X$ for $R = (\mathcal{I}_\bullet, E, F)$, we have that $\Pi^*(M_F(\mathcal{I}_\bullet)) = M_{F'}\Pi^*(\mathcal{I}_\bullet)$.

6. Outline of the algorithm

The main theorem will be a consequence of the following refinement:

Definition (6.1). A *basic object* $\overline{I}$ is a quadruple $(X, \mathcal{I}_\bullet, E, F = \sum_j F^j)$ where X is a smooth scheme of finite type over a field of characteristic zero and $R = (\mathcal{I}_\bullet, E, F)$ is a Rees triple, that is, E is an unordered normal crossings divisor and F is an ordered snc divisor such that $E+F$ is a normal crossings divisor.

Theorem (6.2). *There is a blow-up sequence functor $\mathcal{BQ}$ defined on all basic objects $\overline{I} = (X, \mathcal{I}_\bullet, E, F = \sum_j F^j)$. It satisfies the following conditions:*

(i) *The blow-up sequence $\Pi = \mathcal{BQ}(X, \mathcal{I}_\bullet, E, F)\colon X_r \to X$ is $\overline{I}$-admissible.*
(ii) $\mathcal{BQ}(X, \mathcal{I}_\bullet, E, F) = \mathcal{BQ}(X, \mathcal{J}_\bullet, E, F)$ *if $\mathcal{I} \sim_F \mathcal{J}$.*
(iii) $\operatorname{Supp}(\Pi^*\mathcal{I}_\bullet) = \emptyset$.
(iv) Π *is an isomorphism over $X \setminus \operatorname{Supp}(\mathcal{I}_\bullet)$.*
(v) Π *is functorial with respect to smooth morphisms and change of fields.*
(vi) Π *is functorial with respect to closed embeddings if $E = F = \emptyset$.*

The main theorem is obtained by taking $\mathcal{BP}(X, \mathcal{I}, E) = \mathcal{BQ}(X, (\mathcal{I}, 1), E, \emptyset)$. The algorithm giving Theorem 6.2 consists of the following steps:

(i) If $\dim(X) = 0$, then $V(\mathcal{I})$ is smooth and we are done after the single blow-up with center $\operatorname{Supp}(\mathcal{I}_\bullet)$. By induction we may thus assume that $\mathcal{BQ}$ is constructed for triples $(X, \mathcal{I}_\bullet, E)$ with $\dim(X) < n$. Pick a triple $(X, \mathcal{I}_\bullet, E)$ with $\dim(X) = n$.
(ii) If $\operatorname{Supp}(\mathcal{I}_\bullet) = \emptyset$ we are done.
(iii) If $\mathcal{I}_\bullet$ is a simple Rees algebra (this means that $\operatorname{maxord} \mathcal{I}_\bullet = 1$), then we replace $\mathcal{I}_\bullet$ with $D(\mathcal{I}_\bullet)$. After doing this, there is, using that we are in characteristic zero, a smooth hypersurface of maximal contact $H \in \mathcal{I}_1$ (locally in the Zariski topology). We first perform a boundary modification that makes $E + H$ a snc divisor. Then we proceed by the blow-up sequence $\mathcal{BQ}(H, \mathcal{I}_\bullet|_H, E|_H)$. Włodarczyk's trick proves that this blow-up sequence does not depend on the chosen H.
(iv) If $\mathcal{I}_\bullet$ is not a simple Rees algebra, i.e., $\operatorname{maxord}(\mathcal{I}_\bullet) > 1$, then $\mathcal{I}_\bullet \sim \mathcal{I}^\curlyvee \cdot \mathcal{I}_E$ where $\mathcal{I}_E$ is an exceptional Rees algebra. We then proceed by induction on the weak order $\omega = \operatorname{maxord}_{x \in \operatorname{Supp}(\mathcal{I}_\bullet)} \mathcal{I}_\bullet^\curlyvee$. The weak order is a rational positive number but has bounded denominator.
(v) If the weak order is zero, then $\mathcal{I}_\bullet$ is exceptional and we can resolve the Rees algebra by blowing up various intersections of the exceptional divisors. Here it is crucial that the E_i's are ordered.
(vi) If the weak order is positive, then we let $\mathcal{J}_\bullet = \mathcal{I}_\bullet + (\mathcal{I}^\curlyvee)^{1/\omega}$. Then $\mathcal{J}_\bullet$ is a simple Rees algebra and we can eliminate $\mathcal{J}_\bullet$ by step (iii). After this elimination, the weak order drops and we are done.

7. Restriction to smooth hypersurfaces (maximal contact)

In general, the support of a marked ideal or a Rees algebra does not commute with restrictions. This is however the case if the Rees algebra is differentially closed (or the marked ideal is D-balanced in Kollár's notation):

Lemma (7.1). *Let $H \hookrightarrow X$ be a smooth subvariety and let $\mathcal{I}_\bullet$ be a Rees algebra. Then* $\operatorname{Supp}(\mathcal{I}_\bullet) \cap H \subseteq \operatorname{Supp}(\mathcal{I}_\bullet|_H)$*. If $\mathcal{I}_\bullet$ is differentially closed, then equality holds.*

Proof. It is easily verified that if $x \in H$, then $\operatorname{ord}_x(\mathcal{I}) \leq \operatorname{ord}_x(\mathcal{I}|_H)$. This gives the inclusion. If $\mathcal{I}_\bullet$ is differentially closed, then we have that $\operatorname{Supp}(\mathcal{I}_\bullet) = \operatorname{Supp}(\mathcal{I}_1)$. As $\operatorname{Supp}(\mathcal{I}_\bullet|_H) \subseteq \operatorname{Supp}(\mathcal{I}_1|_H) = \operatorname{Supp}(\mathcal{I}_1) \cap H$ the equality follows. $\square$

Proposition (7.2). *Let $\mathcal{I}_\bullet$ be a Rees algebra and let $H \hookrightarrow X$ be a smooth hypersurface such that* $\operatorname{Supp}(\mathcal{I}_\bullet) \subseteq H$*, e.g., $\mathcal{I}_H \subseteq \mathcal{I}_1$. If $D(-\log H)(\mathcal{I}_\bullet) = D(\mathcal{I}_\bullet)$ then*

(i) $\operatorname{Supp}(\mathcal{I}_\bullet) = \operatorname{Supp}(\mathcal{I}_\bullet|_H)$

(ii) *If $\pi\colon \operatorname{Bl}_Z X \to X$ is a blow-up in a smooth center $Z \hookrightarrow X$ such that $Z \subseteq \operatorname{Supp}(\mathcal{I}_\bullet)$ and H' denotes the strict transform of H, then $D(-\log H')(\pi^*\mathcal{I}_\bullet) = D(\pi^*\mathcal{I}_\bullet)$.*

Proof. Recall that $\operatorname{Supp}(\mathcal{I}_\bullet) = \operatorname{Supp}(D(\mathcal{I}_\bullet)) = \operatorname{Supp}(D(\mathcal{I}_\bullet)|_H)$ and that $D(-\log H)(\mathcal{I}_\bullet)|_H = D(\mathcal{I}_\bullet|_H)$. Thus (i) follows.

Clearly $D(-\log H')(\pi^*\mathcal{I}_\bullet) \subseteq D(\pi^*\mathcal{I}_\bullet)$. To see the reverse inclusion, choose local coordinates such that $Z = V(x_1, x_2, \dots, x_r)$ and $H = V(x_r)$. It is enough to look at the charts $D_+(x_i)$ with $i = 1, 2, \dots, r-1$ since $D = D_H$ on the last chart. We have that $D(\mathcal{I}_m, m) = D(-\log H)(\mathcal{I}_m, m) + (\partial_{x_r}\mathcal{I}_m, m-1)$ and by assumption $(\partial_{x_r}\mathcal{I}_m, m-1) \in D(-\log H)(\mathcal{I}_m, m)$. Similarly, we have that

$$\begin{aligned}
D\pi^*(\mathcal{I}_m, m) &= D(-\log H')\pi^*(\mathcal{I}_m, m) + \partial_{y_r}\pi^*(\mathcal{I}_m, m) \\
&= D(-\log H')\pi^*(\mathcal{I}_m, m) + \pi^*(\partial_{x_r}\mathcal{I}_m, m-1) \\
&\subseteq D(-\log H')\pi^*(\mathcal{I}_m, m) + \pi^* D(-\log H)(\mathcal{I}_m, m) \\
&\subseteq D(-\log H')\pi^*(\mathcal{I}_m, m)
\end{aligned}$$

where we have used equations (4.1.1) when passing between the first and second row. $\square$

Remark **(7.3).** The proof above is much shorter than Kollár [Kol07, Thm. 3.88] as he does not use an inductive assumption.

Corollary (7.4). *Let $\mathcal{I}_\bullet$ be a simple D-closed Rees algebra (or merely D_E-closed). Let $H \hookrightarrow X$ be a smooth hypersurface such that $\mathcal{I}_H \subseteq \mathcal{I}_1$. Then pushing forward from H to X gives an equivalence between blow-up sequences for $(H, \mathcal{I}_\bullet|_H)$ and blow-up sequences for $(X, \mathcal{I}_\bullet)$.*

If in addition $E \hookrightarrow X$ is an snc divisor such that $H + E$ is snc, then pushing forward from H to X gives an equivalence between blow-up sequences for $(H, \mathcal{I}_\bullet|_H, E|_H)$ and blow-up sequences for $(X, \mathcal{I}_\bullet, E)$.

Proof. Note that trivially $D(-\log H)(\mathcal{I}_\bullet) = D(\mathcal{I}_\bullet) = \mathcal{I}_\bullet$. Thus, by the previous proposition, $\operatorname{Supp}(\mathcal{I}_\bullet|_H) = \operatorname{Supp}(\mathcal{I}_\bullet)$ so we have an equivalence for blow-up sequences of length 1.

Let $\pi\colon X' \to X$ be a blow-up with smooth center Z contained in $\operatorname{Supp}(\mathcal{I}_\bullet)$. Let $\mathcal{I}'_\bullet = \pi^*\mathcal{I}_\bullet$ and let H' be the strict transform of H. Then $\pi^*H = H' + E$ is snc and if locally $H = V(x_r)$ so that $(x_r, 1) \in \mathcal{I}_1$, then $H' = V(y_r)$ and $(y_r, 1) = \pi^*(x_r, 1) \in \mathcal{I}'_1$. In particular, $\operatorname{Supp}(\mathcal{I}'_\bullet) \subseteq H'$.

By the previous proposition we have that $D(-\log H)(\pi^*\mathcal{I}_\bullet) = D(\pi^*\mathcal{I}_\bullet)$ and $\mathrm{Supp}(\pi^*\mathcal{I}_\bullet) = \mathrm{Supp}(\pi^*\mathcal{I}_\bullet|_H)$. We may thus conclude by induction.

If we are given an snc divisor E such that $E + H$ is snc, then a smooth center $Z \hookrightarrow H$ has simple normal crossings with E if and only if it has simple normal crossings with $E + H$. If this happens, then $E' + H'$ is snc where E' denotes the total transform of E. □

8. Uniqueness of maximal contact

Proposition (8.1) (Włodarczyk)**.** *xxx*

9. The algorithm

The algorithm consists of three different steps:

(i) The maximal contact case where the hypersurface of maximal contact is transversal to E. (Here we replace $\mathcal{I}_\bullet$ with a D-closed ideal.)

(ii) The maximal contact case where the hypersurface of maximal contact is not necessary transversal to $E+F$. Then we make a boundary modification.

(iii) The general case which goes by induction on the weak order ω_F. The trivial case $\omega_F = \infty$, is taken care by a single blow-up. To reduce ω_F, one resolves a certain companion Rees algebra which is in the maximal contact case. Finally when $\omega_F = 0$, we are in the monomial case.

REMARK: Below is an attempt to make things better that didn't work out. One has to be more clever to get functoriality with respect to smooth morphisms when $E + F \neq \emptyset$.

Definition (9.1). Let $E \hookrightarrow X$ be a normal crossings divisor. Let $X_{E,s} = \{x \in X \ : \ \mathrm{ord}_x(E) = s\}$. This gives a stratification $X = \coprod_{s=1}^n X_{E,s}$ of X into locally closed smooth subvarieties. Define $s(x)$ such that $x \in X_{E,s(x)}$. Let $\mathcal{I}_\bullet$ be a Rees algebra on X. We let

$$\sigma_{\mathcal{I}_\bullet,E}(x) = \dim_{k(x)}(\mathcal{I}_1)_x/(\mathcal{I}_{X_{E,s(x)}} + \mathfrak{m}_x^2).$$

It can be seen that $\sigma_{\mathcal{I}_\bullet,E}$ is a lower semi-continuous function. The integer $m = \sigma_{\mathcal{I}_\bullet,E}$ is the maximum number of smooth hypersurfaces $H_1, H_2, \dots, H_m$ containing $V(\mathcal{I}_1)$, locally around x, such that $H_1 \cap H_2 \cap \cdots \cap H_m$ is a smooth subvariety of codimension m meeting E transversely at x. Such H_i's are hypersurface of maximal contact and $H_1 \cap H_2 \cap \cdots \cap H_m$ acts as a codimension m subvariety of maximal contact.

Theorem (9.2). *Let $m \leq n$ be integers. Assume that $\mathcal{BQ}$ has been constructed for all triples $(X, \mathcal{I}_\bullet, E, F)$ such that $\dim(X) \leq n - m$. There is a blow-up sequence functor $\mathcal{BQ}_m$ defined on all basic objects $(X, \mathcal{I}_\bullet, E, F = \sum_j F^j)$ such that*

- $\dim(X) \leq n$,
- $\mathrm{maxord}(\mathcal{I}_\bullet) \leq 1$, *and*
- $\sigma_{\mathcal{I}_\bullet,F}(x) \geq m$ *for all* $x \in \mathrm{Supp}(\mathcal{I}_\bullet)$.

The blow-up functor $\mathcal{BQ}_m$ has the same properties as $\mathcal{BQ}$ in Theorem (6.2). In particular, it commutes with closed immersions.

Proof. If $m=0$, then we let $\mathcal{BQ}_0 = \mathcal{BQ}$.

Case $E=\emptyset$: Since $m \geq 1$, there exists $f \in (\mathcal{I}_1)_x$ such that $H_x = V(f)$ is a smooth hypersurface meeting H transversally. We may thus find an open neighborhood $x \in U_x$ and lift f to a section of $(\mathcal{I}_1)(U_x)$ such that $H_x = V(f)$ is a smooth hypersurface meeting F transversally. Choose $H_x \subseteq U_x$ for every $x \in X$ and let $X' = \coprod U_x$ for some finite open subcovering. For any other choice of open covering and local hypersurfaces, there is a common refined covering X'' with $f_1, f_2 \in \Gamma(X'', \mathcal{I}_1)$.

It is thus enough to construct $\mathcal{BQ}_m$ for basic objects $(X, \mathcal{I}_\bullet, \emptyset, F = \sum_j F^j)$ as in the theorem under the additional assumption that there exists a global section $f \in \Gamma(\mathcal{I}_1)$ such that $H = V(f)$ is a smooth hypersurface meeting F transversally. However, our construction of $\mathcal{BQ}_m$ is not allowed to depend on the choice of f.

Note that, since F and H meet transversally, $F|_H$ is snc. We let $\mathcal{BQ}_m(X, \mathcal{I}_\bullet, \emptyset, F) = \mathcal{BQ}_{m-1}(H, D_F(\mathcal{I}_\bullet)|_H, \emptyset, F|_H)$. This blow-up sequence is independent of the choice of H by Section 8.

By construction, $\mathcal{BQ}_m$ commutes with smooth morphisms. Indeed, D_F commutes with smooth pull-back and $H \hookrightarrow X$ is pulled back to a hypersurface of maximal contact defined by an element $f \in \mathcal{I}_1$.

If $\mathcal{I} \sim_F \mathcal{J}$, then $D_F(\mathcal{I}_\bullet) \sim_F D_F(\mathcal{J}_\bullet)$ but it need not be possible to find one H that works for both $\mathcal{I}$ and $\mathcal{J}$ (in particular, it could happen that $\mathcal{I}_1 \neq \mathcal{J}_1$). However, we have that $\mathcal{I} \sim_F \mathcal{I} + \mathcal{J} \sim_F \mathcal{J}$, so we can assume that $\mathcal{I} \subseteq \mathcal{J}$. Then we may choose H from $\mathcal{I}$ and it also works for $\mathcal{J}$. Then $\mathcal{I}_\bullet|_H \sim_F \mathcal{J}_\bullet|_H$.

Case $E \neq \emptyset$: In this case, the hypersurface of maximal contact H constructed in the previous case need not meet $E+F$ transversally. We will therefore first do a "boundary" blow-up sequence to make $\mathrm{Supp}(\mathcal{I}_\bullet)$ disjoint from E. There will be new exceptional divisors added to F in the process but these will all be transverse to H.

Note that $\mathrm{maxord}(\mathcal{I}_E) \leq n$, i.e., E has at most n-fold intersections. We begin with resolving $\mathcal{I}_\bullet$ along n-fold intersections. This means looking at the Rees algebra $\mathcal{I}_\bullet + (\mathcal{I}_E, n)$ since $\mathrm{Supp}(\mathcal{I}_\bullet + (\mathcal{I}_E, n)) = \mathrm{Supp}(\mathcal{I}_\bullet) \cap \mathrm{Supp}(\mathcal{I}_E, n)$. Now take the blow-up sequence

$$\Pi_n = \mathcal{BQ}_m(X, \mathcal{I}_\bullet + (\mathcal{I}_E, n), \emptyset, F) \colon X^{(n)} \to X,$$

let $F^{(n)} \subseteq X^{(n)}$ denote the exceptional divisor, let $E^{(n)} = \Pi_n^* E$ be the strict transform and let $\mathcal{I}_\bullet^{(n)} = \Pi_n^* \mathcal{I}_\bullet$ be the transform. Then $\emptyset = \mathrm{Supp}(\Pi_n^*(\mathcal{I}_\bullet + (\mathcal{I}_E, n))) = \mathrm{Supp}(\mathcal{I}_\bullet^{(n)}) \cap \mathrm{Supp}(\Pi_n^*(\mathcal{I}_E, n))$. This means that $\mathrm{maxord}_{x \in \mathrm{Supp}(\mathcal{I}_\bullet^{(n)})} \mathcal{I}_E^{(n)} < n$, i.e., $\mathrm{Supp}(\mathcal{I}_\bullet^{(n)})$ does not meet n-fold intersections of $E^{(n)}$.

Also, for any local hypersurface $H = V(f)$ given by $f \in \mathcal{I}_1$ with $\mathrm{ord}_x(f) = 1$ we have that $\mathrm{Supp}(\mathcal{I}_\bullet) \subseteq H$ and this holds after any number of admissible blow-ups (where we transform H as the strict transforms). This means that $H^{(n)} := \Pi_n^* H = V(f^{(n)})$ where $(f_1, 1) = \Pi_n^*(f, 1) \subseteq \mathcal{I}_1^{(n)}$ and $H^{(n)}$ is transverse to $F^{(n)}$. Thus, we have the blow-up sequence:

$$\Pi_{n-1} = \mathcal{BQ}_1(X^{(n)}, \mathcal{I}_\bullet^{(n)} + (\mathcal{I}_E, n-1), F^{(n)}) \colon X^{(n-1)} \to X.$$

We let $F^{(n-1)} \subseteq X^{(n-1)}$ denote the exceptional divisor, which, by definition, includes $\Pi_{n-1}^{*}F^{(n)}$. Again, we let $E^{(n-1)} = \Pi_{n-1}^{*}E^{(n)}$ be the strict transform and let $\mathcal{I}_{\bullet}^{(n-1)} = \Pi_{n-1}^{*}\mathcal{I}_{\bullet}^{(n)}$ be the transform. We proceed in this way and obtain a blow-up sequence $\Pi = \Pi_n \circ \Pi_{n-1} \circ \Pi_{n-2} \circ \cdots \circ \Pi_0 : X^{(0)} \to X$. We let $\mathcal{BQ}(X, \mathcal{I}_{\bullet}, E) = \Pi$. The only thing we have to verify is that all centers have simple normal crossings with the strict transforms of E. This follows by the construction since in the sequence Π_k, all centers are contained in k-fold intersections of E and $\operatorname{Supp}(\mathcal{I}_{\bullet}^{(k+1)})$ does not meet E in any $k+1$-fold intersection. □

Corollary (9.3). *Assume that $\mathcal{BQ}$ has been constructed for all triples $(X, \mathcal{I}_{\bullet}, E, F)$ such that $\dim(X) < n$. There is a blow-up sequence functor $\mathcal{BQ}_{MC}$ defined on all basic objects $(X, \mathcal{I}_{\bullet}, E, F = \sum_j F^j)$ such that*

- $\dim(X) \leq n$, *and*
- $\operatorname{maxord}(\mathcal{I}_{\bullet}) \leq 1$.

The blow-up functor $\mathcal{BQ}_{MC}$ has the same properties as $\mathcal{BQ}$ in Theorem (6.2)*. In particular, it commutes with closed embeddings when $E = F = \emptyset$.*

Proof. Take $\mathcal{BQ}_{MC}(X, \mathcal{I}_{\bullet}, E, F) = \mathcal{BQ}_1(X, D(\mathcal{I}_{\bullet}), E + F, \emptyset))$. Note that if $(\mathcal{I}_{\bullet}, E, F) \sim (\mathcal{I}'_{\bullet}, E', F)$, then $(\mathcal{I}_{\bullet}, E + F, \emptyset) \sim (\mathcal{I}'_{\bullet}, E' + F, \emptyset)$ and thus also $(D(\mathcal{I}_{\bullet}), E + F, \emptyset) \sim (D(\mathcal{I}'_{\bullet}), E' + F, \emptyset)$. □

Remark **(9.4).** Kollár has a slightly different algorithm which requires that $E = \sum_j E^j$ is snc and ordered. In our terminology, the blow-up sequence for $(X, \mathcal{I}_{\bullet}, E, \emptyset)$ would start with the blow-up sequences

$$\mathcal{BQ}_1(E_i, D(\mathcal{I}_{\bullet})|_{E_i}, \sum_{j \neq i} E^j | E_i, \emptyset)$$

for $i = 1, 2, \ldots, r$ and then finally

$$\mathcal{BQ}_1(X, D(\mathcal{I}_{\bullet}), \emptyset, F)$$

where F is the exceptional divisors given by the blow-ups along the E_i's.

Proof of Theorem (6.2). If $X = \emptyset$, then there is nothing to prove. Fix an integer n and assume that $\mathcal{BQ}$ has been constructed for all quadruples $(X, \mathcal{I}_{\bullet}, E, F)$ such that $\dim(X) \leq n$.

Trivial case $\omega = \infty$: Let $Z = \{x \in X \; : \; \operatorname{ord}_x \mathcal{I}_{\bullet} = \infty\}$. Then Z is the disjoint union of the connected components of X that are contained in $\operatorname{Supp}(\mathcal{I}_{\bullet})$. We begin by blowing up Z. This replaces X and Z with $X \setminus Z$ and $\emptyset$.

Companion case $\omega > 0$: Let $\Delta = \Delta(\mathcal{I}_{\bullet}) \in \mathbb{Z}_{\geq 1}$ be the lcm of the degrees of local generators of $\mathcal{I}_{\bullet}$. Next, let $\omega = \omega_{\mathcal{I}_{\bullet}, F} = \operatorname{maxord}_{x \in \operatorname{Supp}(\mathcal{I}_{\bullet})} N_F(\mathcal{I}_{\bullet})$. This is be maximal weak order of $\mathcal{I}_{\bullet}$ with respect to F. Recall that $\omega \in \frac{1}{\Delta}\mathbb{N}$ and that this remains valid after transformations (§5.9).

If $\omega > 0$, then let $\mathcal{J}_{\omega} = \mathcal{I}_{\bullet} + N_F(\mathcal{I}_{\bullet})^{1/\omega}$. We then perform the blow-up sequence $\mathcal{BQ}_{MC}(X, \mathcal{J}_{\omega}, E + F, \emptyset)$. Since $\mathcal{I}_{\bullet} \subseteq \mathcal{J}_{\omega}$, the sequence is $\mathcal{I}_{\bullet}$-admissible. After replacing $(X, \mathcal{I}_{\omega}, E, F)$ with the transform along this sequence, we have that $\operatorname{Supp}(\mathcal{I}_{\bullet}) \cap \operatorname{Supp}(N_F(\mathcal{I}_{\bullet})^{1/\omega}) = \emptyset$, so $\operatorname{maxord} N_F(\mathcal{I}_{\bullet}) < \omega$. We repeat until $\omega = 0$.

Monomial case $\omega = 0$: If $\omega = 0$, then $N_F(\mathcal{I}_\bullet) = \mathcal{O}_X[t]$ is the trivial Rees algebra and $\mathcal{I}_\bullet = M_F(\mathcal{I}_\bullet)$ is monomial.

MONOMIAL CASE

Functoriality and invariance under equivalence: Let us verify that the algorithm does not depend on the F-similarity class of $\mathcal{I}_\bullet$. If $\mathcal{I}_\bullet \sim_F \mathcal{I}'_\bullet$, then $\omega_{\mathcal{I},N} = \omega_{\mathcal{I}',N}$ and $N_F(\mathcal{I}_\bullet) \sim_F N_F(\mathcal{I}'_\bullet)$ so that $\mathcal{J}_\omega \sim_F \mathcal{J}'_\omega$ [insert reference]. For the monomial case, we note that $\mathrm{ord}_{F_i}\mathcal{I}_\bullet = \mathrm{ord}_{F_i}\mathcal{I}'_\bullet$.

To see that the algorithm is functorial with respect to closed embeddings when $E = F = \emptyset$, assume that we have a smooth subvariety $X_0 \hookrightarrow X$, $X_0 \neq X$ such that $V(\mathcal{I}_\bullet) \subseteq X_0$. Then $\mathrm{maxord}\,\mathcal{I}_\bullet = 1$ and $\omega = 1$. The algorithm will thus be equal to $\mathcal{BQ}_{MC}(X, \mathcal{J}_\omega, \emptyset, \emptyset)$ which is functorial with respect to closed embeddings. $\square$

Appendix A. Order of equivalent Rees triples

Theorem (A.1). *Let $R = (\mathcal{I}_\bullet, E, F)$ and $R' = (\mathcal{J}_\bullet, E', F)$ be weakly equivalent Rees triples. Let $x \in \mathrm{Supp}(\mathcal{I}_\bullet) = \mathrm{Supp}(\mathcal{J}_\bullet)$. Then $\mathrm{ord}_x\mathcal{I}_\bullet = \mathrm{ord}_x\mathcal{J}_\bullet$.*

Proof. [2]

By passing to an open neighborhood we can assume that $Z = \overline{x}$ is smooth and has simple normal crossings with $E{+}F$. Let $X_1 = X \times \mathbb{A}1$, $E_1 = X \times \{0\}$, $\Gamma_1 = \pi_1^{-1}(Z)$ and $Z_1 = E_1 \cap \Gamma_1 = Z \times \{0\}$. Let $(\mathcal{I}_\bullet)^{(1)} = \pi_1^*\mathcal{I}_\bullet$. Having constructed (X_i, Γ_i, Z_i), we let

(i) $\pi_{i+1}\colon X_{i+1} = \mathrm{Bl}_{Z_i} X_i \to X_i$,
(ii) $\Gamma_{i+1} \hookrightarrow X_{i+1}$ be the strict transform of Γ_i,
(iii) $E_{i+1} = \pi_{i+1}^{-1}(Z_i)$ be the exceptional divisor,
(iv) $Z_{i+1} = E_{i+1} \cap \Gamma_{i+1}$, and
(v) $(\mathcal{I}_\bullet)^{(i+1)} = \pi_{i+1}^*(\mathcal{I}_\bullet)^{(i)}$.

Let $\mu = \mathrm{ord}_Z\mathcal{I}_\bullet$. Then, locally, there exists $(f_0, d) \in \mathcal{I}_\bullet$ such that $\mathrm{ord}_Z(f_0) = d\mu$. Let $\pi_i^*(f, d) = (f_i, d)$. Pick local coordinates (at any point of Z_i) such that $D_i = (x_1)$, $\Gamma_i = (x_2, x_3, \dots, x_r)$ and $Z_i = (x_1, x_2, \dots, x_r)$. We will prove by induction that $f_i = x_1^{di(\mu-1)} g_i(x_2, \dots, x_n)$ where $\mathrm{ord}_Z(g_i) = d\mu$. This is clear for $i = 1$ as $f_1 = f_0$ only depends on $x_2, x_3, \dots, x_n$. Now, assume that $f_i = x_1^{di(\mu-1)} g_i(x_2, \dots, x_n)$. Then it is only the chart $D_+(x_1)$ that contains Γ_{i+1}. On this chart we have that $D_{i+1} = (y_1)$, $\Gamma_{i+1} = (y_2, y_3, \dots, y_r)$ and

$$
\begin{aligned}
f_{i+1} &= y_1^{-d} f_i(y_1, y_1y_2, y_1y_3, \dots, y_1y_r, y_{r+1}, \dots, y_n) \\
&= y_1^{-d+di(\mu-1)} g_i(y_1y_2, y_1y_3, \dots, y_1y_r, y_{r+1}, \dots, y_n) \\
&= y_1^{-d+di(\mu-1)+d\mu} g_i(y_2, y_3, \dots, y_n) \\
&= y_1^{d(i+1)(\mu-1)} g_i
\end{aligned}
$$

Thus $g_{i+1} = g_i$ has order $d\mu$ at Z_i and the order along D_i is as claimed.

We have thus shown that $\mathrm{ord}_{D_i}(\mathcal{I}_\bullet^{(i)}) = i(\mu - 1)$. In particular, we have that $D_i \subseteq \mathrm{Supp}(\mathcal{I}_\bullet^{(i)})$ if and only if $i(\mu - 1) \geq 1$.

If $i(\mu - 1) \geq 1$, then the blow-up at the divisor D_i is admissible. This is a trivial blow-up in the sense that X_i, Z_i, Γ_i and D_i are left unchanged but $\mathcal{I}_\bullet^{(i)}$ is modified. If we let p denote the blow-up in D_i, then $\mathrm{ord}_{D_i}(p^*\mathcal{I}_\bullet^{(i)}) =$

[2] Clash with E in the proof.

$i(\mu-1)-1$. Similarly, if $i(\mu-1)\geq q$, then we may blow-up q times along D_i and obtains $\mathrm{ord}_{D_i}(p^*)^q\mathcal{I}_\bullet^{(i)}=i(\mu-1)-q$. This sequence of blow-ups is admissible if and only if $i(\mu-1)-q\geq 0$ or equivalently, precisely when

$$\mu\geq 1+\frac{q}{i}.$$

By assumption, $\mathrm{ord}_x\mathcal{I}_\bullet,\mathrm{ord}_x\mathcal{J}_\bullet\geq 1$. Thus, if $\mathrm{ord}_x(\mathcal{I}_\bullet)>\mathrm{ord}_x(\mathcal{J}_\bullet)$, then there exists a sequence of blow-ups that is only admissible for $\mathcal{I}_\bullet$ but not for $\mathcal{J}_\bullet$ and vice versa. □

Theorem (A.2). *Let $R=(\mathcal{I}_\bullet,E,F)$ and $R'=(\mathcal{J}_\bullet,E',F)$ be equivalent Rees triples. Then*

- $\mathrm{ord}_x\mathcal{I}_\bullet=\mathrm{ord}_x\mathcal{J}_\bullet$ *for all* $x\in\mathrm{Supp}(\mathcal{I}_\bullet)=\mathrm{Supp}(\mathcal{J}_\bullet)$.
- $\mathrm{ord}_{F^j}\mathcal{I}_\bullet=\mathrm{ord}_H\mathcal{J}_\bullet$ *for every irreducible component F^j of F.*

Proof. [3] (i) follows from the previous theorem.

For (ii), we will proceed as in (i) with $Z=F^j$. The difference is that $\mathrm{ord}_{F^j}\mathcal{I}_\bullet<1$ is possible and then the blow-ups are not admissible. However, E_i and Γ_i are exceptional divisors so $D_i=E_i\cap\Gamma_i$ is an exceptional center and we can use exceptional blow-ups.

Let $\mu=\mathrm{ord}_H\mathcal{I}_\bullet$. We obtain $\mathrm{ord}_{D_i}(\mathcal{I}_\bullet^{(i)})=i\mu$. As before, if $i\mu\geq q$, we may then do an admissible blow-up q times along D_i and obtain a blow-up sequence which is a test sequence if and only if $i\mu\geq q$, or equivalently, if and only if $\mu\geq\frac{q}{i}$. Thus, $\mathrm{ord}_H\mathcal{I}_\bullet=\mathrm{ord}_H\mathcal{J}_\bullet$. □

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[3] Clash with E in the proof.