Mini-course: Resolution of singularities

2013-03-15

Lecture #3

Infinitely near singularities

Termination of algorithm: normal forms

Termination of algorithm: maximal contact

Termination of algorithm: Weierstrass preparation

Infinitesimally near singularities (§1.5, 1.40)

For simplicity, assume $S = \mathbb{A}^2_k$. Then $C \subset S$ given by an equation $f(x,y)=0$. Assume that C singular at the origin w/ multiplicity m. Then:

$$f(x,y) = f_m(x,y) + f_{m+1}(x,y) + \cdots$$

We blow-up in the origin: $\pi : Bl_0 S \longrightarrow S$

Chart $D_+(x)$: $k[x,y,\frac{x}{x},\frac{y}{x}] \cong k[x,\frac{y}{x}]$ $\quad x_1 = x, \; y_1 = \frac{y}{x} \quad\quad x = x_1, \; y = x_1 y_1$

$$(\pi^* f)(x_1,y_1) = f(x_1, x_1 y_1) = x_1^m f_m(1,y_1) + x_1^{m+1} f_{m+1}(1,y_1) + \cdots$$
$$= x_1^m \left(f_m(1,y_1) + x_1 g \right) = x_1^m \tilde{f}$$

(and similarly for $D_+(y)$)

Thus: (Lemma 1.40)

a) $\pi^{-1} C = mE + \tilde{C}$

b) $\tilde{C} \cap E \hookrightarrow E \cong \mathbb{P}^1$ is given by $f_m = 0$. In particular $|\tilde{C} \cap E| = m$ when counted w/ multiplicity.

c) $\sum_{p \in \tilde{C} \cap E} \mathrm{mult}_p \tilde{C} \leq \mathrm{mult}_0 C = m$

The singularities $p \in \tilde{C} \cap E$ are called <u>infinitely near singularities</u> in the first infinitesimal neighborhood of $0 \in C$.

Proof: a) Note that $x_1 \nmid \tilde{f} \Rightarrow \tilde{C} = V(\tilde{f})$

b) First chart: $\tilde{C} \cap E = V(f_m(1,y_1)) \hookrightarrow \mathbb{A}^1 = \mathrm{Spec}(k[y_1])$, $y_1 = y/x$

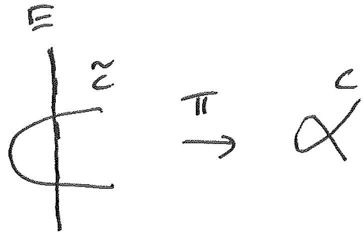
Second chart: $\tilde{C} \cap E = V(f_m(x_1,1)) \hookrightarrow \mathbb{A}^1 = \mathrm{Spec}(k[x_1])$, $x_1 = x/y$

$\Big\}$ $\tilde{C} \cap E = V(f_m(x,y)) \hookrightarrow \mathbb{P}^1$

c) As $\mathrm{mult}_p \tilde{C} \leq \mathrm{mult}_p \tilde{C} \cap E$ we have:

$$\sum_p \mathrm{mult}_p \tilde{C} \leq \sum_p \mathrm{mult}_p (\tilde{C} \cap E) = m.$$

In particular, if $\tilde{C} \cap E$ has more than 1 point, then the multiplicity drops.
This happens exactly when $f_m \neq c \cdot l^m$ where $l(x,y)$ is a linear form, $c \in k$.
(if k is perfect).



Ex: $f = \underbrace{y^2 - x^2}_{f_2} - x^3 = (y-x)(y+x) - x^3$

two distinct pts of $\tilde{C} \cap E$

Ex: $f = y^a - x^b$ $\quad m = a < b, \quad f_m = y^m$

So $\tilde{C} \cap E$ has one point $\underset{p}{\wedge}$ of multiplicity m

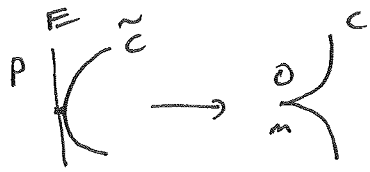
no drop in mult.

On the chart $D_+(x)$ we have

$$f = (x_1 y_1)^a - x_1^b = x_1^a (y_1^a - x_1^{b-a})$$

$$\tilde{f} = y_1^a - x_1^{b-a}$$

If $b - a \geq a = m$, then $\mathrm{mult}_p \, \tilde{C} \cap E = m$, otherwise the multiplicity drops to $b-a$.



drop in mult.

In this example, it's obvious that the multiplicity eventually drops.

Termination of embedded resolution of curves algorithm

Normal forms (2a) (Kollar §1.8)

For simplicity, assume $S = \mathbb{A}^2_k$ (k perfect as before).

It is enough, by induction, to prove that there is no infinite blow-up sequence:

$S_{i+1} = Bl_{p_i} S_i$

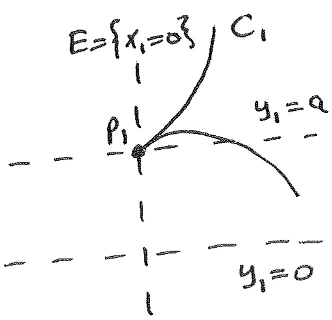
$C_{i+1} = Bl_{p_i} C_i$

$$\begin{tikzcd}
\cdots \arrow[r] & S_2 \arrow[r] & S_1 \arrow[r] & S_0 = S \\
 & C_2 \arrow[r] \arrow[u, hook] \arrow[d, phantom, sloped, "\ni"] & C_1 \arrow[r] \arrow[u, hook] \arrow[d, phantom, sloped, "\ni"] & C_0 \arrow[u, hook] \arrow[d, phantom, sloped, "\ni"] \\
 & p_2 & p_1 & p_0
\end{tikzcd}$$

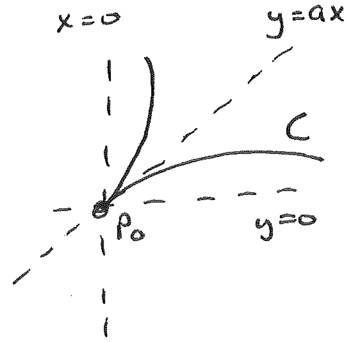
where $\operatorname{mult}_{p_i} C_i = m \ \forall i$.

WLOG, $k = \bar{k}$ and $p_0 = (0,0)$.

Let $f(x,y) = f_m + \cdots$. We have seen that the multiplicity drops unless $f_m = c\,l^m$ where $l = y - a_1 x$ is a linear form.



$y_1 = \frac{y}{x}$



If we do the coordinate change

$(x, y) \longmapsto (x, y - a_1 x)$

then on the first chart we have the coordinate change

$(x_1, y_1) \longmapsto (x_1, y_1 - a_1)$

so that p_1 is ~~at~~ the origin.

Similarly, if the multiplicity does not drop under the second blow-up, there is a coordinate change:

$(x_1, y_1) \longmapsto (x_1, y_1 - a_2 x)$

such that p_2 lies at the origin of the second blow-up. We may thus do the composite coord. change:

$(x, y) \longmapsto (x, y - a_1 x - a_2 x^2)$

to ensure that p_0, p_1, p_2 all lie at the origins.

Similarly, if $\mathrm{mult}_{p_k} C_k = m$, then there is a $\overset{\text{unique}}{\vee}$ coord change:

$$(x,y) \longmapsto (x, y - a_1 x - a_2 x^2 - \dots - a_k x^k) = (x, y_k)$$

s.th. $p_0, p_1, \dots, p_k$ all lie at the origins and then

$$f \in (x^{k+1}, y_k)^m$$

If there's an <u>infinite</u> sequence, then in $k[[x,y]]$ we have

$$f \in (x^{k+1}, y_\infty)^m \quad \forall k \geq 0 \quad \text{where } y_\infty = y - a_1 x - a_2 x^2 - \dots$$

By Krull's intersection theorem:

$$f \in \bigcap_{k \geq 0} (x^{k+1}, y_\infty)^m \subset \bigcap_{k \geq 0} (x^{k+1}, y_\infty^m) = (y_\infty^m)$$

This gives a contradiction: $V(f) = C \hookrightarrow \mathbb{A}^2_k$ has an isolated singularity at the origin but $V(f) \hookrightarrow \mathrm{Spec}(k[[x,y]])$ is singular (non-reduced) along $y_\infty = 0$. To see that this is impossible, we note that

$$k[[x,y]]/ f, \tfrac{\partial f}{\partial x}, \tfrac{\partial f}{\partial y} = \left(k[x,y]/ f, \tfrac{\partial f}{\partial x}, \tfrac{\partial f}{\partial y}\right)^\wedge = k[x,y]/ f, \tfrac{\partial f}{\partial x}, \tfrac{\partial f}{\partial y}$$

is zero-dimensional, but

$$\left(f, \tfrac{\partial f}{\partial x}, \tfrac{\partial f}{\partial y}\right) = \left(f, \tfrac{\partial f}{\partial x}, \tfrac{\partial f}{\partial y_\infty}\right) \subset (y_\infty^{m-1})$$

is one-dimensional. (also ok if $m = p = \mathrm{char}\, k$, then get $\subset (y_\infty^m)$)

<u>Rmk</u>: This proof works for any <u>excellent</u> regular surface S such that the singular points have perfect residue fields. The crucial point is that $\hat{\mathcal{O}}_{C,p}$ is reduced (b/c $\hat{S} \to S$ geom. regular fibres).

Hypersurfaces of maximal contact

(2b) (Kollar §1.10)

"Find the optimal hypersurface $y_\infty = 0$"

<u>Def:</u> $\overset{p\in}{C} \hookrightarrow S$, $\mathrm{ord}_p C = m$. A smooth curve $p \in H \hookrightarrow S$ is a curve of <u>maximal contact</u> of C at p if after any seq of blow-ups in smooth centers $\pi : S' \longrightarrow S$ then H' contains every point $p' \in \pi^{-1}(p)$ s.th. $\mathrm{mult}_{p'} C' \geq m$.
$\uparrow$ equiv $=$

Here C' and H' denote the strict transforms of C and H.

<u>Rmk:</u> • Note that H' is a curve of maximal contact of C' at all points above p.

• If we pick local coordinates at p s.th. $p = (0,0)$ and $H = (y=0)$ then $p_i \in C_i$ lies at the origin of the first chart for all i. Indeed, $H_i = (y_i = 0)$.

<u>Thm 1.81:</u> Let $p \in C \hookrightarrow S$, $H \hookrightarrow S$, m be as above. After at most $\left\lfloor \frac{M}{m} \right\rfloor$ blow-ups, where $M = (C \cdot H)_p$, every point $p' \in C'$ above p has multiplicity $< m$.

<u>Pf:</u> Locally, choose coordinates s.th. $H = (y=0)$. Then

$$f = y^m + y \cdot g(x,y) + x^M \cdot h(x) \qquad h(0) \neq 0$$

After k blow-ups, we have in the distinguished chart:

$$f_k = y_k^m + y_k \cdot g_k(x, y_k) + x^{M-mk} \cdot h(x) \qquad H_k = (y_k = 0) \qquad y_k = \frac{y}{x^k}$$

$$C_k = (f_k = 0) \qquad g_k(x, y_k) = \frac{g(x, x^k y_k)}{x^{k(m-1)}}$$

The multiplicity drops below m when either $M - mk < m$ or $g_k(x, y_k) \notin (x, y_k)^{m-1}$.

Existence of hypersurfaces of maximal contact

Let $D = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$ and $D^k = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)^k = \left(\frac{\partial^k}{\partial x^k}, \frac{\partial^k}{\partial x^{k-1}\partial y}, \dots, \frac{\partial^k}{\partial y^k}\right)$.

If $I \subset k[x,y]$ is an ideal, then we let:

$$D(I) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)_{f \in I} \quad \text{and} \quad D^k I = D^{k-1}(D(I)) = \left(\frac{\partial^k f}{\partial x^k}, \dots, \frac{\partial^k f}{\partial y^k}\right)_{f \in I}$$

If $I = (f_1, f_2, \dots, f_s)$ then $D I = \left(f_i, \frac{\partial f_i}{\partial x_j}\right)_{\substack{1 \le i \le s \\ 1 \le j \le n}}$
$\subset k[x_1, \dots, x_n]$

Thm 1.82: S smooth surface over k, $\mathrm{char}(k=0)$, $C \hookrightarrow S$ reduced curve given by ideal $I = (f)$ locally at $p \in C$. If $\mathrm{mult}_p C = m$ then there exists $g \in D^{m-1} I$ s.th. $H = V(g)$ is a smooth curve of maximal contact of C at p. In fact, any $g \in D^{m-1} I$ s.th. $H = V(g)$ is smooth is ——— " ———.

pf. Note that since $\mathrm{mult}_p C = m$, $D^m I = (1)$ locally at p. Thus, $\exists g \in D^{m-1} I$ s.th. $\frac{\partial g}{\partial x}(0) \neq 0$ or $\frac{\partial g}{\partial y}(0) \neq 0$ so that $H = V(g)$ is smooth at p. We will show that H is a curve of maximal contact.

Note that H contains all points of order $\geq m$ (b/c $f \in m^k \Leftrightarrow D(f) \subset m^{k-1}$)

Also, if $S' \xrightarrow{\pi} S$ is a blow-up (in a point), then the strict transform H' is smooth.

It is thus enough to prove that if $g \in D^{m-1} I$, then $g' \in D^{m-1} I'$ where $H' = V(g')$ and $C' = V(f')$, $I' = (f')$.

An easy computation (chain rule) shows that if $\pi : S' \to S$ is a blow-up in p and $E = (x_1 = 0)$ then $x_1^{-(m-1)} \pi^{-1}(D(f)) \subseteq D(x^{-m} \pi^{-1}(f))$ and $x_1^{-1} \pi^{-1}(D^{m-1}(f)) \subseteq D^{m-1}(x^{-m} \pi^{-1}(f))$

so $g' = x_1^{-1} \pi^{*}(f) \in D^{m-1} I' = D^{m-1}(f') = D^{m-1}(x^{-m} \pi^{-1}(f))$.

Weierstrass preparation theorem

(26') Kollár 1.89

$0 \in C \hookrightarrow S$, $\mathrm{mult}_0 C = m$. Choose coordinates such that y^m appears w/ non-zero in the equation $f=0$ for C. Then, by Weierstrass preparation theorem (1.88), we can write the equation in $\hat{\mathcal{O}}_{S,0}$ as: (uniquely)

$$(f) = \left(y^m + g_{m-1}(x) y^{m-1} + \dots + g_0(x) \right)$$

If $m \nmid \mathrm{char}(k)$ then we can do the coordinate change $y' = y - \frac{1}{m} g_{m-1}$ and obtain the simpler form:

$$f' := y'^m + a_{m-2}(x) y'^{m-2} + a_{m-3}(x) y'^{m-3} + \dots + a_0(x) = 0$$

$(f = f' \cdot \text{unit})$

Note that $y' = \frac{\partial^{m-1} f'}{\partial y'^{m-1}}$ so $y'=0$ is a curve of maximal contact.

Thus, after a blow-up we only have to look at the chart $D_+(x)$:

$$\tilde{f}' = x_1^{-m} \pi^{-1}(f') = y_1^m + x_1^{-2} a_{m-2}(x_1) y_1^{m-2} + x_1^{-3} a_{m-3}(x_1) y_1^{m-3} + \dots + x_1^{-m} a_0(x_1)$$

$$\begin{cases} x_1 = x \\ y_1 = y'/x \end{cases} \qquad \begin{cases} x = x_1 \\ y' = x_1 y_1 \end{cases}$$

$\tilde{f}'$ is thus of the same form as f' with $a_k(x) \rightsquigarrow x_1^{k-m} a_k(x_1)$

Note that $\cancel{\mathrm{mult}}_0(f) = m \iff \mathrm{mult}_0(a_{m-h}(x)) \geq k \quad \forall h = 2,3,\dots,m$.

As $\mathrm{mult}_0(a_{m-h}(x))$ decreases w/ k for each blow-up we see that

$$\mathrm{mult}_0 C_{\cancel{k}} = m \iff \cancel{k}\cancel{k} < \left\lfloor \frac{\mathrm{mult}_0(a_{m-h}(x))}{k} \right\rfloor \qquad \forall h = 2,3,\dots,m$$