Mini-course: Resolution of singularities

2013-03-08

Lecture #2

(Resolutions of singularities on curves)

Albanese method

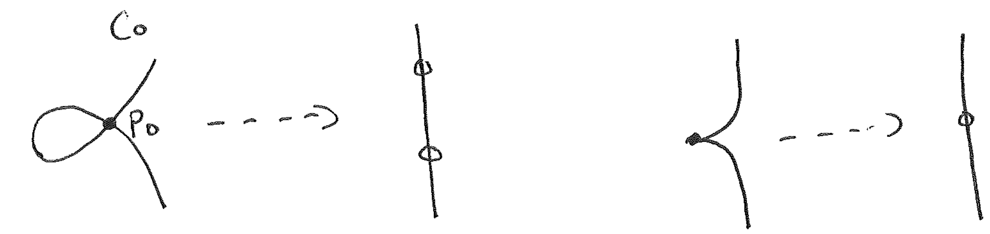
Embedded resolution of curves via blow-ups (on smooth surfaces)

Blow-ups

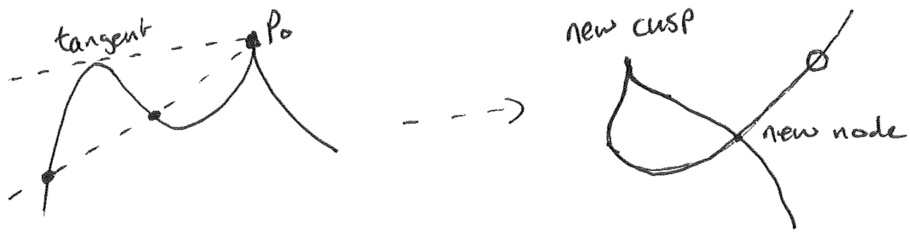
Albanese method (Kollár §1.3)

Algorithm: Let $C_0 \subset \mathbb{P}^n$ be a projective curve. Given $C_i \subset \mathbb{P}^{n-i}$ we pick any singular point $p_i \in C_i$ and let $\pi_i : \mathbb{P}^{n-i} \dashrightarrow \mathbb{P}^{n-i-1}$ be the projection from the point p_i and $C_{i+1} = \overline{\pi_i(C_i)}$.

Ex:



Projecting may create new singularities!



This happens if $\exists$ line through p_0 meeting C_0 in 3 pts (counted w/ multiplicity). Does not occur if $n \geq 4$ and p_0 in general position. But singular points are not necc. in general position!

$k = \bar{k}$ (or k perfect)

Thm (Albanese 1924) Let $C_0 \subset \mathbb{P}^n$ be an irr red proj curve spanning $\mathbb{P}^n_k$ and such that $\deg C_0 < 2n$. Then the algorithm stops with a smooth curve $C_i \subset \mathbb{P}^{n-i}$ which is birational to C_0. The inverse map $C_i \dashrightarrow C_0$ is a morphism so gives a resolution $C_i \longrightarrow C_0$.

Rmk: Such embeddings exist: if L is an ample line bundle then $h^0(C, L^{\otimes m}) = \deg L^{\otimes m} + \chi(C)$ so for $m \gg 0$, $\deg L^{\otimes m} < 2(h^0(C, L^{\otimes m}) - 1)$. The embedding $|L^{\otimes m}|$ for $m \gg 0$ thus works.

Sketch of pf: Prove that projecting from a singular point gives a birational map and that $\deg C_1 < 2(n-1)$, and $C_1 \subset \mathbb{P}^{n-1}$ spanning. Then repeat.

Rmk: There exists a higher-dimensional version of the Albanese method. (Kollár §2.5) It gives $X \underset{\text{birational}}{\dashrightarrow} X'$ such that every point of X' has multiplicity $\leq (\dim X)!$

In char p, points of multiplicity $< p$ are simpler (no wild ramification).
Abhyankar used this to resolve singularities of three-dim. varieties when $p > 3! = 6$.

Embedded resolution of curves

Algorithm: Let S_0 be a smooth surface (over a perfect field) and $C_0 \subset S_0$ a curve. Given $C_i \subset S_i$, pick any singular point $p_i \in C_i$, let $S_{i+1} = Bl_{p_i} S_i \xrightarrow{\pi_i} S_i$ and let $C_{i+1} \subset S_{i+1}$ be the strict transform (so that $C_{i+1} = Bl_{p_i} C_i$).

Thm: The algorithm stops with a smooth curve $C_i \subset S_i$.

Various proofs that algorithm terminates:

1) Global methods: e.g. show that various intersection numbers strictly decrease and are bounded below.

$C_i \cdot (C_i + K_{S_i}) = \deg \omega_C$

(Max Noether, Kollár §1.6)

2) Local methods:

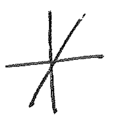
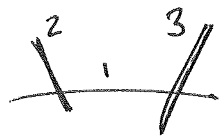
a) Make clever choice of coordinates and prove that an infinite blow-up sequence is impossible. (work on complete local ring) Kollár §1.8

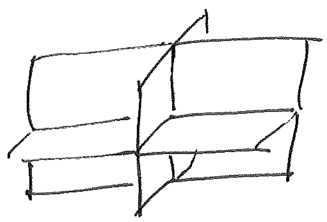
b) Hypersurface of maximal contact (char zero): Choose an "optimal" coordinate system from the beginning. Kollár §1.10

b') Use Weierstrass preparation thm (over complete local ring) to put the equation (char zero) in a simple form where the hypersurface of maximal contact is obvious and the whole resolution process is transparent. Kollár §1.89

Strong embedded resolution of curves

Def: Let X be a smooth variety and $D \subset X$ a divisor. We say that D is a simple normal crossings divisor (snc) if every component of D is smooth and all intersections are transverse, i.e., locally, there $\exists$ $m = (x_1, x_2, \dots, x_n)$ a regular sequence s.th. $D = \{\prod x_i^{m_i} = 0\}$ for some $m_i \in \mathbb{N}$.

Ex:  $\subset \mathbb{A}^2$ not snc 

snc  snc

$xy(x-y) = 0$

$xyz = 0$

 nc but not snc

Algorithm: Let $C_0 \subset S_0$ be a curve in a smooth surface (over a perfect field).
Given $C_i \subset S_i$, pick any point $p_i \in C_i$ where C_i is not snc. Let $S_{i+1} = Bl_{p_i} S_i \xrightarrow{\pi_i} S_i$ and $C_{i+1} \subset S_{i+1}$ be the total transform (i.e. $C_{i+1} = \pi_i^{-1}(C_i)$).

Thm: The algorithm stops with a snc divisor $C_i \subset S_i$.

Rmk: An snc-divisor $D \subset X$ gives a log-smooth pair (X, D).

Proof: Run weak res. algorithm. Get $C_i = \tilde{C}_i + E_i$ where $\tilde{C}_i$ smooth and E_i snc. They can intersect badly though. Run algorithm again on $C_i \subset S_i$. Obtain $C_j \subset S_j$ with $C_j = \tilde{C}_j + F_j$ and $\tilde{C}_j$ smooth, F_j snc. Now, however, every blow-up has center transversal to $\tilde{C}_i$ and E_i so F_j is transverse to $\tilde{C}_j \Rightarrow C_j$ snc.

Final step: Show that any other blow-up sequence also works.

Blow-ups

Def: $Z \hookrightarrow X$ closed subscheme. $Bl_Z X$ or $Bl_I X = \mathrm{Proj}\left(\bigoplus_{d\geq 0} I^d\right)$
defined by ideal sheaf I

Rmks:
- Z — center of blow-up
- $\bigoplus I^d$ — Rees algebra
- $\pi: Bl_Z \to X$ is proper and birational: $\pi|_{X\setminus Z}$ is an iso.
- Exc. divisor $E := \pi^{-1}(Z)$ is a Cartier divisor.
- $X\setminus Z \cong \pi^{-1}(X\setminus Z)$ is sch. dense in $Bl_Z X$.

More rmks:
- If $Z \hookrightarrow X$ is a Cartier divisor, then $Bl_Z X \to X$ iso.
  We still call $E = \pi^{-1}(Z)$ "exceptional".
- If $Z = \emptyset$, then $Bl_Z X \to X$ iso and $E = \emptyset$.

Def: Let $W \hookrightarrow X$ be a second closed subscheme. The strict transform of W along $Bl_Z X \xrightarrow{\pi} X$ is the closed subscheme $\tilde{W} \hookrightarrow Bl_Z X$ given as the closure of $W \cap (X\setminus Z)$ in $Bl_Z X$ $(\supset X\setminus Z)$.

Rmk:
- $\underset{\text{strict trf}}{\tilde{W}} \hookrightarrow \underset{\text{total trf}}{\pi^{-1}(W)}$ and this is an iso outside Z.
- $\tilde{W} \cong Bl_{Z\cap W} W$

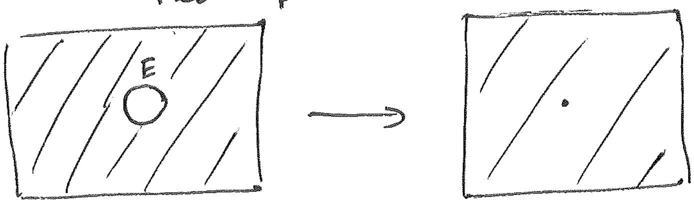
$$\begin{tikzcd}
\tilde{W} \cong Bl_{Z\cap W} W \arrow[r, hook] \arrow[dr] & Bl_Z X \times_X W \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & Bl_Z X \arrow[d, "\pi"] \\
& W \arrow[r, hook] & X
\end{tikzcd}$$

Fact: $\underset{\text{regular}}{Z} \hookrightarrow \underset{\text{regular}}{X}$, then $Bl_Z X$ regular and $E \to Z$ is a $\mathbb{P}^{c-1}$-bundle where $c = \mathrm{codim}(Z \hookrightarrow X)$. Indeed, $E = \mathbb{P}_Z(\mathcal{I}/\mathcal{I}^2)$ ← conormal bundle, rk c

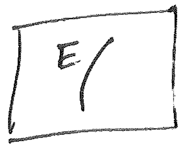
Motivating pictures for blow-ups

real picture

complex picture

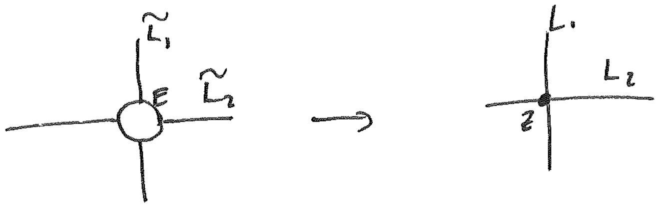


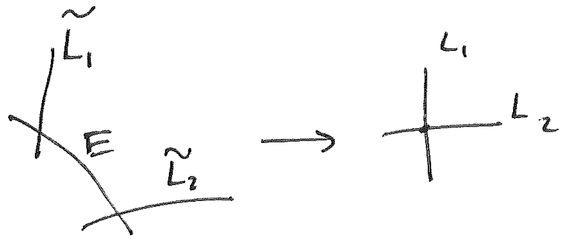
$E = \mathbb{P}^1_{\mathbb{C}}$

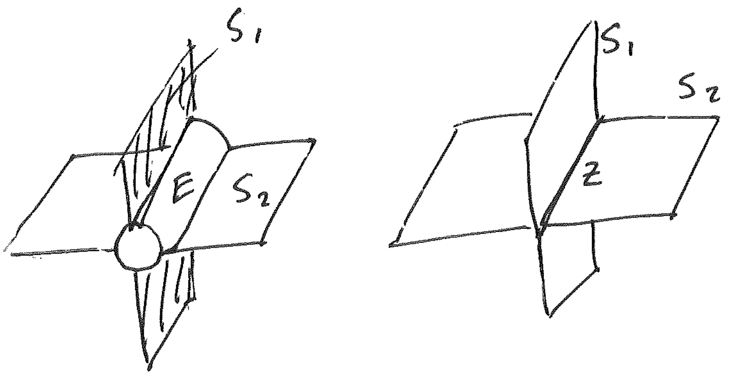


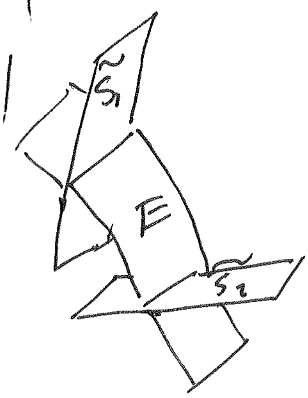
$E = \mathbb{RP}^1 \longrightarrow *$

= projectivization of tangent space









E = projectivization of normal bundle of Z.

Blow-ups and basechange

<u>Rmk</u>: If $X' \xrightarrow{g} X$ <u>flat</u>, then $Bl_{g^{-1}(Z)} X' \xrightarrow{\cong} (Bl_Z X) \times_X X'$.

Indeed $\left(\bigoplus_{d\geq 0} I^d\right) \otimes_{\mathcal{O}_X} \mathcal{O}_{X'} \xrightarrow{\cong} \bigoplus_{d\geq 0} I^d \mathcal{O}_{X'}$.

<u>Rmk</u>: If (A, m) local ring and $(\hat{A}, \hat{m})$ completion, then

- $Gr_m A = Gr_{\hat{m}} \hat{A}$
- $e(A) = e(\hat{A})$ (multiplicity)
- A regular $\Leftrightarrow$ $\hat{A}$ regular
- A/I regular $\Leftrightarrow$ $\hat{A}/I\hat{A} = \widehat{A/I}$ regular.

Thus if $Z \hookrightarrow Spec(A) = X$ are regular, then $\hat{Z} \hookrightarrow Spec(\hat{A}) = \hat{X}$ are regular

and $Bl_{\hat{Z}} \hat{X} \xrightarrow{\cong} (Bl_Z X) \times_X \hat{X}$.

Similarly, if $X' \xrightarrow{g} X$ is <u>smooth</u> and X regular, $Z \hookrightarrow X$ regular, then

$Bl_{g^{-1}Z} X' \xrightarrow{\cong} (Bl_Z X) \times_X X'$ and $g^{-1}Z$ is smooth.

This allows us to study singularities and blow-ups in smooth centers smooth-locally and after passing to completions.

<u>Note</u>: This works more generally for any flat morphism w/ geom. regular fibers.

Explicit charts of blow-ups

$Z \hookrightarrow X = \mathrm{Spec}(A)\,, \quad I = (f_1, f_2, \dots, f_r)$

Every element of I gives a chart:

$$D_+(f) \subset Bl_Z X : \qquad D_+(f) = \mathrm{Spec}\Big(\big(\bigoplus_{d \geq 0} I^d\big)_{(f)}\Big) = \mathrm{Spec}\Big(\overset{A_f \,\cup}{A\left[\tfrac{f_1}{f}, \tfrac{f_2}{f}, \dots, \tfrac{f_r}{f}\right]}\Big)$$

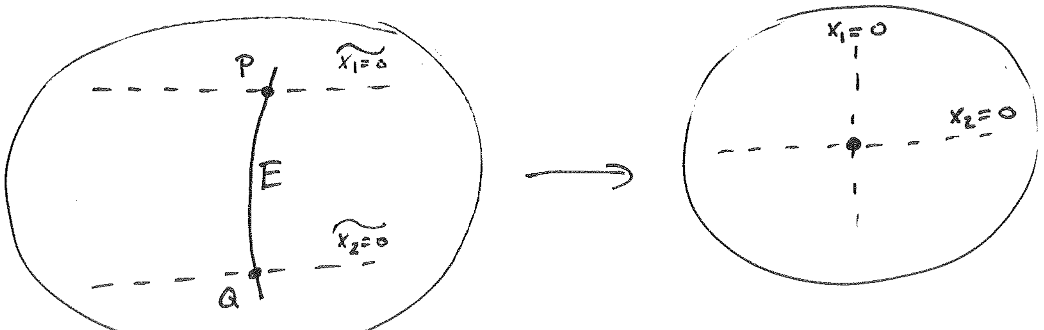
$f \in I^1$

<u>Ex</u>: $A = k[x_1, x_2]$, $I = (x_1, x_2)$

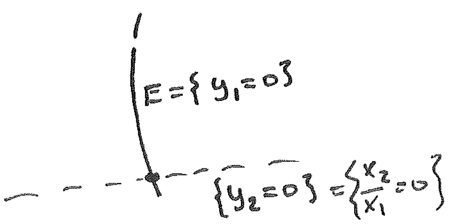
$D_+(x_1)$: $k\left[x_1, x_2, \frac{x_1}{x_1}, \frac{x_2}{x_1}\right] \cong k\left[x_1, \frac{x_2}{x_1}\right]$ $\qquad y_1 = x_1$, $y_2 = \frac{x_2}{x_1}$ $\qquad x_1 = y_1$, $x_2 = y_2 y_1$

exceptional divisor E given by ideal $(x_1, x_2) = y_1$

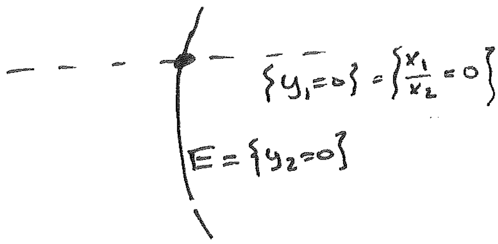
$D_+(x_2)$: $k\left[x_2, \frac{x_1}{x_2}\right]$ $\qquad y_1 = \frac{x_1}{x_2}$, $y_2 = x_2$ $\qquad x_1 = y_1 y_2$, $x_2 = y_2$ $\qquad E$ given by y_2



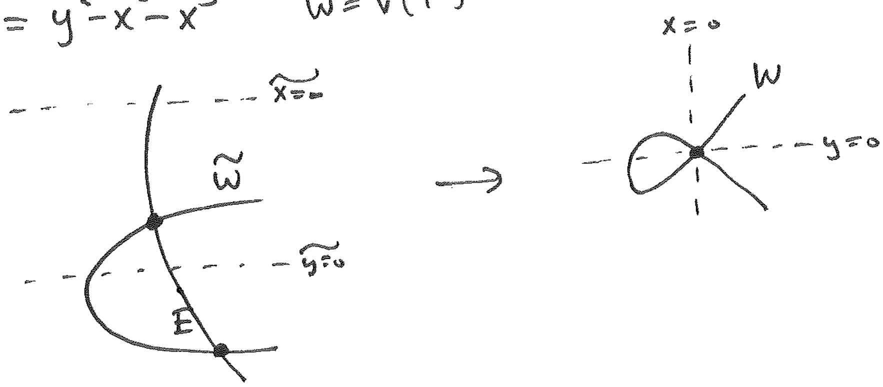
First chart $D_+(x_1)$ misses $\widetilde{x_1=0}$.



Second chart $D_+(x_2)$ misses $\widetilde{x_2=0}$



Ex: $f = y^2 - x^2 - x^3$ $\quad W = V(f)$



First chart:

$$x = x' \qquad x' = x$$
$$y = y'x' \qquad y' = \frac{y}{x}$$

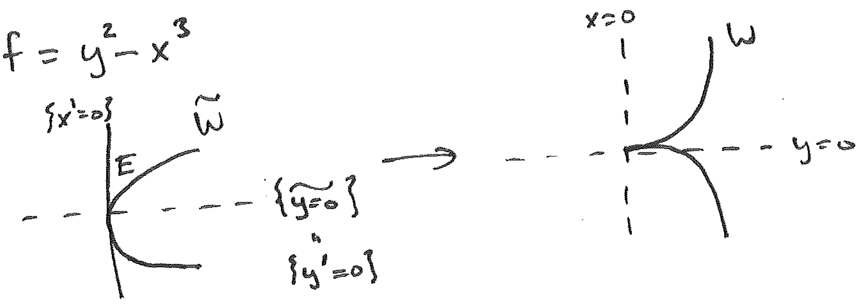
$$f = y'^2x'^2 - x'^2 - x'^3$$
$$= \underbrace{x'^2}_{\text{2 copies of the exc div.}}\underbrace{(y'^2 - 1 - x')}_{=\tilde{f} = \text{ideal of strict tr } \widetilde{W}.}$$

$\pi^{-1}(W) = 2E + \widetilde{W}$

2 copies of the exc div.

Ex: $f = y^2 - x^3$

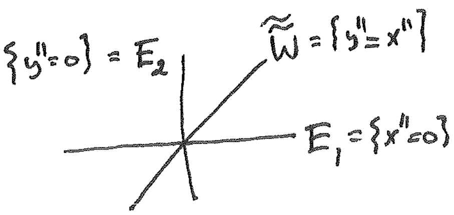
1st Chart



$$f = y'^2x'^2 - x'^3 = x'^2(y'^2 - x')$$

$\pi^{-1}(W) = 2E + \widetilde{W}$

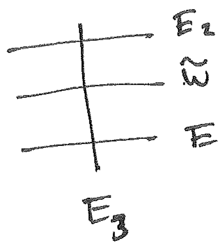
We do a <u>second</u> blow-up in $x' = y' = 0$. This time the chart $D_+(y')$ is relevant



$$f = x''^2y''^2(y''^2 - x''y'') = x''^2y''^3(y'' - x'')$$

$$\pi_2^{-1}\pi_1^{-1}(W) = 2E_1 + 3E_2 + \widetilde{\widetilde{W}}$$

A third blow-up gives



Now, an <u>snc divisor</u>.