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Proof of main thm

Desc of stacky Artin algebraization

Thm (Alper–Hall–R '14) $\mathcal{X}$ alg. stack of f.t./k. $x \in \mathcal{X}(k)$ closed point, w/ affine stabilizer groups w/ stabilizer G_x (group scheme /k). Assume G_x linearly reductive.

$$\begin{array}{c} \mathrm{Spec}\, A \\ \downarrow \\ [\mathrm{Spec}\, A/G_x] \xrightarrow{\text{étale}} \mathcal{X} \\ \rotatebox{90}{$\in$}\quad\quad \rotatebox{90}{$\in$} \\ w \longmapsto x \\ \downarrow \\ \mathrm{Spec}\, A^{G_x} \end{array}$$

"$\mathcal{X}$ is étale-locally a quotient stack"

char 0	char p
GL_n, SL_n	fin grps $p \nmid \lvert G \rvert$
fin grps	$\mathbb{G}_m^n$
$\mathbb{G}_m$	μ_{p^r}
$\vdots$	

Ex: ① $\mathbb{A}^1 \to [\mathbb{A}^1/\mathbb{G}_m] \to *$

② $[\mathbb{A}^2/SL_2]$ $\quad$ $B\mathbb{G}_m$, BSL_2, $B\mathbb{G}_a$

③ $[\alpha/\mathbb{G}_m] \xleftarrow{\text{ét.}} [X/\mathbb{G}_m] = B\mathbb{G}_m$

Ingredients

I Complete local stacks "Grothendieck existence"

G lin red $\circlearrowright A$. Suppose A^G complete local, and fixes $\mathfrak{m} \subset A$ maximal. Then

$$\begin{array}{ccc} \mathrm{Coh}^G(A) & \xrightarrow{\cong} & \varprojlim \mathrm{Coh}^G(A/\mathfrak{m}^n) \\ \| & & \| \\ \mathrm{Coh}(\mathcal{X}) & \xrightarrow{\cong} & \varprojlim \mathrm{Coh}(\mathcal{X}^{[n]}) \end{array}$$

$\mathcal{X} = [\mathrm{Spec}\, A/G]$

$\mathcal{X}^{[n]} = [\mathrm{Spec}(A/\mathfrak{m}^n)/G]$

II Tannaka duality

$$\begin{aligned} \mathrm{Hom}(\mathcal{X}, \mathcal{Y}) &= \mathrm{Hom}_\otimes(\mathrm{Coh}(\mathcal{Y}), \mathrm{Coh}(\mathcal{X})) \\ &= \mathrm{Hom}_\otimes(\mathrm{Coh}(\mathcal{Y}), \varprojlim \mathrm{Coh}(\mathcal{X}^{[n]})) \\ &= \varprojlim \mathrm{Hom}_\otimes(\mathrm{Coh}(\mathcal{Y}), \mathrm{Coh}(\mathcal{X}^{[n]})) \\ &= \varprojlim \mathrm{Hom}(\mathcal{X}^{[n]}, \mathcal{Y}) \end{aligned}$$

$$\boxed{\mathcal{X} = \varinjlim \mathcal{X}^{[n]}}$$

III Artin algebraicity (equivariant version)

Artin approximation [Art69a]

A henselian local excellent.

$A \longrightarrow \hat{A}$ completion.

$$(*) \begin{cases} f_1(x_1, \dots, x_n) = 0 \\ \vdots \\ f_m(x_1, \dots, x_n) = 0 \end{cases} \qquad f_j \in A[x_1, \dots, x_n]$$

sol to $(*)$ in $\hat{A}$

$$\Updownarrow$$

$$B = A[x_i]/(f_j) \xrightarrow{\bar{\alpha}} \hat{A}$$

<u>Artin/Popescu</u>: ('69) $\forall c \geq 0$ $\exists\, B \xrightarrow{\alpha} A$ s.th. $\hat{\alpha} : B \longrightarrow A \longrightarrow \hat{A}$

Given $B \xrightarrow{\bar{\alpha}} \hat{A}$ $\qquad \bar{\alpha} \equiv \hat{\alpha} \pmod{c}$

$$\begin{tikzcd}
 & & \operatorname{Spec}(B) \\
\operatorname{Spec}(A/m^c) \arrow[r, hook] \arrow[urr] & \operatorname{Spec}(\hat{A}) \arrow[r] \arrow[ur, "\bar{\alpha}"] & \operatorname{Spec}(A) \arrow[u, dashed, "\alpha", "\exists"']
\end{tikzcd}$$

s.th. big diagram commute.

Functorial point of view

Replace $\operatorname{Spec} B$ w/ functor $F : Sch^{op} \longrightarrow Set$.

<u>Notation</u>: $\underset{\underset{Sch}{\rotatebox{90}{$\in$}}}{X} \xrightarrow{x} F \quad = \quad x \in F(X)$

<u>Ex</u>: $F = \operatorname{Hom}_{Sch}(-, Y) = h_Y$

<u>Def</u>: F loc fin. pres: $\varinjlim F(\operatorname{Spec} A_\lambda) \xrightarrow{\simeq} F(\operatorname{Spec}(\varinjlim A_\lambda))$ $\quad \{A_\lambda\}$ direct system of k-alg.

<u>Ex</u>: h_Y l.f.p $\Leftrightarrow$ Y l.f.p. $/k$.
$\quad = $ l.f.t.

<u>Thm</u> (<u>AA</u>): F l.f.p. $\quad c \in \mathbb{N}$

$$\begin{tikzcd}
 & & F \\
\operatorname{Spec}(A/m^c) \arrow[r, hook] \arrow[urr] & \operatorname{Spec} \hat{A} \arrow[r] \arrow[ur] & \operatorname{Spec} A \arrow[u, dashed, "\exists"']
\end{tikzcd}$$

s.th. big diagram commutes.

Pf:

$$\begin{tikzcd}
 & & & F \\
 & & \operatorname{Spec} B \arrow[ur] & \\
 \cdot \arrow[urr] \arrow[r, hook] & \cdot \arrow[ur] \arrow[r] & \cdot \arrow[u, dashed] &
\end{tikzcd}$$

Appl 1: Formal vs algebraic equivalence. TFAE

(Artin '69) $\underset{x}{X} \ni$, $\underset{y}{Y} \ni$ f.t. / k. (1) $\hat{\mathcal{O}}_{X,x} \cong \hat{\mathcal{O}}_{Y,y}$

(2) $\exists$

$$\begin{tikzcd} & (W,w) \arrow[dl, "\text{ét}"'] \arrow[dr, "\text{ét}"] & \\ (X,x) & & (Y,y) \end{tikzcd}$$

pf:

$$\begin{tikzcd} \operatorname{Spec} \hat{\mathcal{O}}_{X,x} \arrow[r, "\cong"] \arrow[d] & \operatorname{Spec} \hat{\mathcal{O}}_{Y,y} \arrow[dd] \\ \operatorname{Spec} \mathcal{O}^h_{X,x} \arrow[d] \arrow[dr, dashed, "\varphi^h"] \arrow[r, phantom, "\text{①}"] & \\ \operatorname{Spec} \mathcal{O}_{X,x} \arrow[d] & Y \\ X & \end{tikzcd}$$

① Artin approximation (w/ $c = 2$).

$\hat{\varphi}^h \equiv \bar{\varphi}$ (mod m^2)

$\Rightarrow \hat{\varphi}$ isomorphism (but not equal to $\bar{\varphi}$!)

② Easy approx.

$$\begin{tikzcd} & W \arrow[dl] \arrow[dr, "\varphi"] & \\ \operatorname{Spec} \mathcal{O}_{X,x} & & Y \end{tikzcd}$$

$\hat{\varphi}$ iso $\Rightarrow \varphi$ étale.

Appl 2

~~Ex~~: Algebraicity of isolated singularities

$\hat{A} = k[[x_1, \dots, x_n]] \twoheadrightarrow \bar{B}$

$A = k[x_1, \dots, x_n]^h$

<u>Thm</u> (Artin '69)

Assume $\mathrm{Spec}(\bar{B})$ form. smooth outside origin.

Then $\exists\ A \twoheadrightarrow B$ s.th. $\hat{B} \cong \bar{B}$.

NB: B unique by Étale equivalence.

pf: Resolve $\hat{A}^{n_2} \xrightarrow{\bar{\varphi}_2} \hat{A}^{n_1} \xrightarrow{\bar{\varphi}_1} \hat{A} \twoheadrightarrow \bar{B} \to 0$

Approximate $A^{n_2} \xrightarrow{\varphi_2} A^{n_1} \xrightarrow{\varphi_1} A$

s.th. $\underbrace{\boxed{\varphi_1 \circ \varphi_2 = 0}}_{\text{eqn!}}$ $\underset{\text{not exact}}{(\text{complex})}$ and $\hat{\varphi}_1 \equiv \bar{\varphi}_1 \pmod c$, $\hat{\varphi}_2 \equiv \bar{\varphi}_2 \pmod c$

Define $B = \mathrm{coker}\ \varphi_1$.

Hironaka '68: For $c \gg 0$, $\hat{B} \cong \bar{B}$.

<u>Whitney '65</u> $\exists$ analytic ~~subvariety~~ hypersurface $V \subset \mathbb{C}^3$ s.th. $\mathcal{O}_{V,p}$ not algebraic.

(not isolated sing)
5 smooth $\mathbb{C}^2$
meeting in a line

normal analytic hypersurface $W \subset \mathbb{C}^4$
s.th. $\mathcal{O}_{W,p}$ not algebraic.

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Artin algebraization [Art69b]

Question: When is a complete local k-alg. $k[[x_1,\dots,x_n]]/I = \overline{B}$ the completion of B fin. type $/k$?

Magic trick: Formal versality.

Thm (Artin '69) F functor l.f.p. Given form versal $\operatorname{Spec}\overline{B} \xrightarrow{\eta} F$ then $\exists$:

$$\begin{tikzcd} & F & \\ \operatorname{Spec}\overline{B} \arrow[ur, "\eta"] \arrow[rr, dashed] \arrow[rr, phantom, "\eta \not\cong \xi" above] & & \operatorname{Spec}B \arrow[ul, "\xi"'] \end{tikzcd}$$

form. versal, form versal, $\exists$ $\quad B$ f.t.$/k$.

Def:
$$\begin{tikzcd} \operatorname{Spec}\overline{B} \arrow[r, "\eta"] & F \\ \operatorname{Spec}R \arrow[u] \arrow[r, hook] & \operatorname{Spec}R' \arrow[u] \arrow[ul, dashed, "\exists"'] \end{tikzcd}$$
form versal means $\forall\, R' \twoheadrightarrow R$ surj of artinian rings sq zero kernel

pf: (Conrad-de Jong '02) $A = k[x_1,\dots,x_n]^h$

$$\hat{A} = k[[x_1,\dots,x_n]] \twoheadrightarrow \overline{B}$$

Choose res. $\hat{A}^{n_2} \xrightarrow{\overline{\varphi}_2} \hat{A}^{n_1} \xrightarrow{\overline{\varphi}_1} \hat{A} \to \overline{B}$. Pick $c \in \mathbb{N}$ s.th.

"Artin-Rees lemma works for c" for $\overline{\varphi}_1$ and $\overline{\varphi}_2$ (AR_c).

Approx. $A^{n_2} \xrightarrow{\varphi_2} A^{n_1} \xrightarrow{\varphi_1} A$ s.th $\varphi_1\varphi_2 = 0$. $\hat{\varphi}_i \equiv \overline{\varphi}_i \mod c$.

Define $B = \operatorname{coker}\varphi_1$. Have

$$\begin{tikzcd} \hat{A}^{n_2} \arrow[r, "\overline{\varphi}_2"] \arrow[d, equal] & \hat{A}^{n_1} \arrow[r, "\overline{\varphi}_1"] \arrow[d, equal] & \hat{A} \arrow[r] \arrow[d, equal] & \overline{B} \\ \hat{A}^{n_2} \arrow[r, "\hat{\varphi}_2"] & \hat{A}^{n_1} \arrow[r, "\hat{\varphi}_1"] & \hat{A} \arrow[r] & \hat{B} \end{tikzcd}$$
commutes mod c

$(AR)_c$ for $\overline{\varphi}_2$ and $\overline{\varphi}_1$ $\Rightarrow$ $(AR)_c$ holds for $\hat{\varphi}_1$ (and bottom complex exact)

$$\Rightarrow \operatorname{gr}_m \overline{B} \cong \operatorname{gr}_m \hat{B}.$$
simple algebra

$$\operatorname{gr}_m \overline{B} = \bigoplus_{n\geq 0} m^n\overline{B}/m^{n+1}\overline{B}$$

Artin alg. (cont)

pf (cont)

$$\begin{tikzcd}
\operatorname{Spec} \overline{B} \arrow[rrr, "\eta"] & & & F \\
\operatorname{Spec} B/m^c \arrow[u, "\psi_c"] \arrow[r, hook] & \operatorname{Spec} B/m^{c+1} \arrow[ul, dotted, "\psi_{c+1}"] \arrow[r, hook] & \cdots \arrow[r, hook] & \operatorname{Spec} \hat{B} \arrow[u] \arrow[lll, dotted, bend right]
\end{tikzcd}$$

$\Rightarrow$

$$\begin{tikzcd}
& F & \\
\operatorname{Spec} \overline{B} \arrow[ur] & \circ & \operatorname{Spec} \hat{B} \arrow[ul] \arrow[ll, "\psi"]
\end{tikzcd}$$

ψ isom mod c by constr. $\Rightarrow$ ψ closed imm.

$gr_m \overline{B} \cong gr_m \hat{B}$ $\Rightarrow$ ψ isom.

not coming from ψ!

$(AR)_c$ for $\varphi : M \xrightarrow{\varphi} N$ A-module homo.

$$\varphi(M) \cap m^n N \subseteq \varphi(m^{n-c} M) \quad \forall n \geq c$$

Artin-Rees Lemma: $\exists c$. s.th $(AR)_c$ holds.

Complete local rings and stacks

(I) A ~~complete~~ local noeth ring.

$$\mathrm{Coh}(A) \xrightarrow{\Psi} \varprojlim_n \mathrm{Coh}(A/m^n)$$

Ψ isom $\iff$ A complete.

(II) Grothendieck's existence thm.

$$X \xrightarrow{p} \mathrm{Spec}(A) \text{ proper } \left(\text{scheme / stack}\right)$$

Then $$\mathrm{Coh}(X) \underset{\cong}{\xrightarrow{\Psi}} \varprojlim_n \mathrm{Coh}(X^{[n]})$$

closed subsch. def by $m\mathcal{O}_X = I$

$$\begin{tikzcd} X^{[0]} = p^{-1}(*) \arrow[r, hook] \arrow[d] & X \arrow[d] \\ * \arrow[r, hook] & \mathrm{Spec}\, A \end{tikzcd}$$

$m^{n+1}\mathcal{O}_X = I^{n+1}$

$$\begin{tikzcd} X^{[n]} \arrow[r, hook] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}\, A/m^n \arrow[r, hook] & \mathrm{Spec}\, A \end{tikzcd}$$

Def: $X^{[0]} \underset{\text{closed}}{\hookrightarrow} X$. Say X coh. complete along $X^{[0]}$ if

$$\mathrm{Coh}(X) \xrightarrow{\cong} \varprojlim_n \mathrm{Coh}(X^{[n]})$$

where $X^{[n]}$ n^{th} inf. nbhd of $X^{[0]}$.

(I): Spec A coh complete along closed pt.

(II) X —— " —— ~~$p^{-1}(*)$~~ closed fiber.

Cons. of Tannaka formalism: (HR'14) X coh complete along $X^{[0]}$. X excellent. Y alg. stack w/ aff. stab.

$$\mathrm{Hom}(X, Y) \xrightarrow{\cong} \varprojlim_n \mathrm{Hom}(X^{[n]}, Y)$$

That is: $$\boxed{X = \varinjlim_n X^{[n]}}$$

1/4 (AHR)

Thm: A noeth ring$/k$. G lin red. group scheme$/k$ $\curvearrowright A$. Suppose A^G complete local. Suppose G fixes m, max ideal. Then

$$\mathrm{Coh}^G(A) \xrightarrow{\simeq} \varprojlim \mathrm{Coh}^G(A/m^n)$$

Stacky int.
$\mathcal{X} = [\mathrm{Spec}(A)/G]$
$\mathcal{X}^{[n]} = [\mathrm{Spec}(A/m^{n+1})/G]$

$$\mathrm{Coh}\,\mathcal{X} \xrightarrow{\simeq} \varprojlim \mathrm{Coh}(\mathcal{X}^{[n]})$$

i.e. $\mathcal{X}$ complete along $\mathcal{X}^{[0]}$.

$$\begin{tikzcd} * \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{Spec}\,A \arrow[d] \\ BG = \mathcal{X}^{[0]} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\text{not cart.}"] & \mathcal{X} \arrow[d] \\ * \arrow[r, hook] & \mathrm{Spec}(A^G) \end{tikzcd}$$

$\mathrm{Spec}\,A$: not complete!
$\mathcal{X}$: complete local stack
$\mathrm{Spec}(A^G)$: complete local ring.

Examples

$[\mathbb{A}^1/\mathbb{G}_m] = (B\mathbb{G}_m \;\cdot) \longrightarrow \cdot = \mathrm{Spec}\,k[t]^{\mathbb{G}_m} = \mathrm{Spec}\,k$

$[\mathbb{A}^2/SL_2] = (BSL_2 \;\; B\mathbb{G}_a) \longrightarrow \cdot$

$[\mathbb{A}^2/\mathbb{G}_m^2] = (B\mathbb{G}_m \;\cdot\; B\mathbb{G}_m^2 \; B\mathbb{G}_m) \longrightarrow \cdot$

$[\mathbb{A}^2/\mathbb{G}_m] = (B\mathbb{G}_m)\;\mathbb{P}^1 \longrightarrow \cdot$

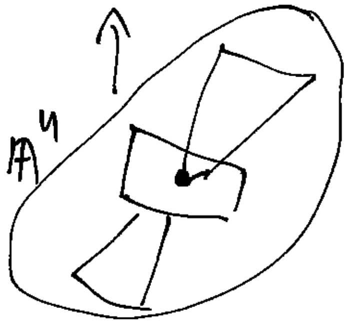
$[\mathbb{P}^1/\mathbb{G}_m] = (B\mathbb{G}_m \;\cdot\; B\mathbb{G}_m)$. no gms.

$[X/\mathbb{G}_m] \to [\alpha/\mathbb{G}_m] = (B\mathbb{G}_m \;\cdot)$ — " —
sing

$[X/\mathbb{G}_m] = (\cdot\; B\mathbb{G}_m \;\cdot)$

$[\mathbb{A}^4/\mathbb{G}_m] = (\mathbb{P}^1 \; \bigcirc B\mathbb{G}_m \bigcirc \; \mathbb{P}^1 \; X^{ns}) \longrightarrow X = \mathrm{Spec}\,k[xz, xw, yz, yw]/(xz)(yw)-(xw)(yz)$

wts $1,1,-1,-1$
x, y, z, w



Applications:

1) Completion at $x \in \mathcal{X}$ exists:

$$\begin{tikzcd} \hat{\mathcal{X}}_x \arrow[r] \arrow[d] & {[\operatorname{Spec} A/G_x]} \arrow[d] \\ \operatorname{Spec} \hat{A}^{G_x} \arrow[r] & \operatorname{Spec} A^{G_x} \end{tikzcd}$$

2) If $\mathcal{X}, \mathcal{Y}$ stacks (as in thm) and $\hat{\mathcal{X}}_x \cong \hat{\mathcal{Y}}_y$ then $\exists$

$$\begin{tikzcd} & (\mathcal{W}, w) \arrow[dl, "\text{ét}"'] \arrow[dr, "\text{ét}"] & \\ (\mathcal{X}, x) & & (\mathcal{Y}, y) \end{tikzcd}$$

"étale-loc. equiv"

More appl:

- Luna
- Sumihiro
- actions on spaces (Drinfeld)
- Białynicki-Birula for Artin stacks
- compact generation of der. categories
- theory of good moduli spaces.
- equivariant miniversal deform. spaces
- toric Artin stacks.
- Grothendieck existence / Formal GAGA

Stacky Artin ~~algebraization~~ approx.

$$\begin{tikzcd} & & F & \\ \mathcal{X}_x^{[n]} \arrow[rru, bend left] \arrow[r, hook] & \hat{\mathcal{X}}_x \arrow[ru] \arrow[rr] \arrow[rru, phantom, "\circ"] & & \mathcal{X}_x^h \arrow[lu, dashed, "\exists"'] \end{tikzcd}$$

straight-fwd from AA.

Stacky Artin algebraization

$$\begin{tikzcd} & F = \mathcal{X} & \\ \hat{\mathcal{X}}_x \arrow[ru, "\text{f.ét.}"] \arrow[rr] \arrow[rd] & & W \arrow[lu, "\text{f.ét.}"'] \\ & \mathcal{X}_x^h \arrow[ru] & \end{tikzcd}$$

Proof of main thm:

Step 1: $\mathcal{X}_x^{[0]} \hookrightarrow \mathcal{X}_x^{[1]} \hookrightarrow \cdots$ (Deformation theory: simple)

Step 2a: $\mathcal{N}_x = [N_x / G_x]$ normal stack.

"effectivize"

$$\begin{tikzcd} \mathcal{X}_x^{[0]} \arrow[r, hook] \arrow[d, "\wr"] & \mathcal{X}_x^{[1]} \arrow[r, hook] \arrow[ld] & \mathcal{X}_x^{[2]} \cdots \arrow[lld] \\ \mathcal{N}_x & & \end{tikzcd}$$

(deformation theory)

$\Rightarrow \exists\, \mathcal{X}_x^\wedge \hookrightarrow \hat{\mathcal{N}}_x$

$\hat{\mathcal{N}}_x$ coh complete

Step 2b: "effectivize" Tannaka duality

$\mathcal{X}_x^\wedge \longrightarrow \mathcal{X}$

Step 3 "algebraize" $\mathcal{X}_x^\wedge \longrightarrow W \xrightarrow{\text{ét.}} \mathcal{X}$