Small-talk

The étale topology

11/11 - 2013

Why étale topology?

X scheme $/\mathbb{C}$, $X^{an} = X(\mathbb{C})$ w/ complex topology.

Many properties that hold analytically (for X^{an}) or formally ($\hat{\mathcal{O}}_{X,x}$) does not hold for Zariski topology.

1) Implicit function theorem

$$\begin{array}{c} X \ni x \\ f \downarrow \\ Y \end{array}$$

f smooth at x, then $f^{an}: X^{an} \to Y^{an}$ has a section locally around x and $f(x) = y$, even: $X^{an} \cong Y^{an} \times \mathbb{C}^n$ locally at x.

formally (over an alg closed field) $\hat{\mathcal{O}}_{X,x} \cong \hat{\mathcal{O}}_{Y,y}[[t_1, \ldots, t_n]]$

Ex: $Y = \mathbb{A}^1 \setminus 0 = \mathrm{Spec}(k[t, t^{-1}])$

$u^2 \uparrow \quad \uparrow$

$u \quad X = \mathbb{A}^1 \setminus 0 = \mathrm{Spec}(k[u, u^{-1}] = k[t, t^{-1}, u]/u^2 - t)$

$X \downarrow Y$ finite ét rk 2

f smooth, but no Zar loc section (b/c $K(X) = k(u)$ has no ~~section~~ retract.)
$K(Y) = k(u^2)$

2) G-torsors

G finite group acting freely on X.

$$\begin{array}{c} G \circlearrowright X \\ \downarrow \\ Y = X/G \end{array}$$

free & transitive on fibers — "should" be a G-torsor

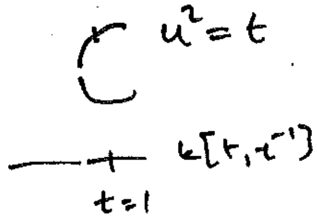
i.e. locally on Y of the form $X = Y \times G$.

Happens if $X \to Y$ has a section loc. on Y.

ok analytically & formally, but not Zar-locally.

Ex above should be a $\mathbb{Z}/2\mathbb{Z}$-torsor.

Ex 1: analytical section



local branch $u = \sqrt{t}$

or $u = -\sqrt{t}$

power series expansion: $u = 1 + \frac{1}{2}(t-1) - \frac{1}{8}(t-1)^2 + \cdots$

$$k[[t-1]] = \widehat{\mathcal{O}}_{X,t-1} \longrightarrow \widehat{\mathcal{O}}_{Y,u-1} = k[[u-1]]$$

$$1 + \tfrac{1}{2}(t-1) + \cdots \longleftarrow\!\shortmid\ u$$

3) irreducible components

X  X irr, X^{an} not irr locally at x

$\widehat{\mathcal{O}}_{X,x} = \widehat{\{X\}} = k[[x,y]]/xy$ not irr.

Zariski topology cannot distinguish this. (irr. but not anal. irr $\Leftrightarrow$ not (geom.) unibranch)
(normalization sees this)

4) cohomology

Want to get algebraic description of $H^1(X^{an}, G) = G$-torsors
$H^n(X^{an}, G)$

Zariski topology insuff. $\pi_1(X^{an})$, $\pi_n(X^{an})$, ...

G-torsors (G finite group)

In (fét) G-torsors are locally trivial:

$$\begin{tikzcd} X\times G \cong X\times_Y X \arrow[r] \arrow[d] & X \arrow[d, "G\text{-torsor}"] \\ X \arrow[u, bend left] \arrow[r, "\text{fét}"'] & Y \end{tikzcd}$$

b/c $X \times_Y X \to X$ has a section.

Ex:
$$\begin{tikzcd} k[v,v^{-1},u,u^{-1}]/u^2-v^2 & k[u,u^{-1}] \arrow[l] & u^2 \\ k[v,v^{-1}] \arrow[u] & k[t,t^{-1}] \arrow[l] \arrow[u] & t \arrow[u, leftrightarrow] \\ v^2 & t \arrow[l, mapsto] & \end{tikzcd}$$

two components: diag $u=v$ $\quad 0\in\mathbb{Z}/2\mathbb{Z}$
antidiag $u=-v$ $\quad 1\in\mathbb{Z}/2\mathbb{Z}$

$H^1_{\text{fét}}(X,G) = H^1(X^{an},G)$ for G finite group.

Fundamental group

(fét) corresponds to covering spaces w/ finite fibers in anal. topology. (# of sheets)

$\pi_1^{\text{ét}}(X,x)$ = pro-finite completion of $\pi_1(X^{an},x)$

$\pi_1^{\text{ét}}(\operatorname{Spec} k) = \operatorname{Gal}_{k^{sep}/k}$ (products of sep. field ext of k corr. to étale alg/k)

For $H^2(X,G)$ and higher fét is not enough. (e.g. $H^2(X,\mu_n)$)

Similarly, for $H^n(X,\mathcal{F})$ fét is not enough (need Zariski covers)

For G special (e.g. GL_n, SL_n, G_m, G_a, solvable) Zar is enough.

So what about topology (Zar) + (fét)? [Grothendieck late 50's?]

Not good enough.

Étale maps

$x \in X$, $y = f(x)$

$X \xrightarrow{f} Y$ finite type, Y noetherian. TFAE

(i) f flat at y + unramified i.e.: $k(y) \longrightarrow \mathcal{O}_{f^{-1}(y),x}$ sep algebra (i.e. sep field extension)

(ii) if $k(y)$ sep closed: $\widehat{\mathcal{O}}_{Y,y} \longrightarrow \widehat{\mathcal{O}}_{X,x}$ isom.

(iii) f formally étale at x (i.e., unique lifting of thickenings)

(iv) f smooth and of rel dim 0 at x

(i.e. smooth and $f^{-1}(y)$ has dim 0 at x)

$\Leftrightarrow$ f q-finite at x

(ét) = topology generated by surj. étale maps.

(fét) = ——"—— surj finite ét maps. (not merely q-fin)

<u>Real thing</u>: Étale topology = (ét).

Has nice <u>local rings</u>: Take $\mathrm{Spec}(\bar{k}) \xrightarrow{x} X$. Then

$^{sh}\mathcal{O}_{X,x} := \mathcal{O}^{\acute{e}t}_{X,\bar{x}}$ = henselization of X at $\bar{x}$ = $x^*_{\acute{e}t}\mathcal{O}_{X_?} = \varinjlim$ étale $\mathcal{O}_{X,x}$-algs. (pointed.)

= localization of integral closure of $\mathcal{O}_{X,x}$ in $\widehat{\mathcal{O}}_{X,x}$ "algebraic power series")

- $^{sh}\mathcal{O}_{X,x}$ irr $\Leftrightarrow$ $\widehat{\mathcal{O}}_{X,x}$ irr
- If $X \to Y$ smooth at $x \in X$, then $\exists$ $Y' \to Y$ étale, $y' \mapsto y$, $s: Y' \to X$ with $s(y') = x$.

$$\begin{tikzcd} & X \arrow[d] \\ Y' \arrow[ur, bend left, "s"] \arrow[r] & Y \end{tikzcd}$$

and $\exists\, U \xrightarrow[\acute{e}t]{} Y \times \mathbb{A}^n \to Y$, $U \overset{\text{open}}{\subset} X$, $x \in U$.

<u>Nisnevich topology</u> (ét) = (fét) + (Nis)