NOETHERIAN APPROXIMATION OF ALGEBRAIC SPACES AND STACKS

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Abstract. We show that every scheme (resp. algebraic space, resp. algebraic stack) that is quasi-compact with quasi-finite diagonal can be approximated by a noetherian scheme (resp. algebraic space, resp. stack). More generally, we show that any stack which is étale-locally a global quotient stack can be approximated. Examples of applications are generalizations of Chevalley's, Serre's and Zariski's theorems and Chow's lemma to the non-noetherian setting. We also show that every quasi-compact algebraic stack with quasi-finite diagonal has a finite generically flat cover by a scheme.

Introduction

Let A be a commutative ring and let M be an A-module. Then A is the direct limit of its subrings that are finitely generated as $\mathbb{Z}$-algebras and M is the direct limit of its finitely generated A-submodules. Thus, any affine scheme X is an inverse limit of affine schemes of finite type over $\operatorname{Spec}\mathbb{Z}$ and every quasi-coherent sheaf on X is a direct limit of quasi-coherent sheaves of finite type.

The purpose of this article is to give similar approximation results for schemes, algebraic spaces and stacks, generalizing earlier results for schemes by R. W. Thomason and T. Trobaugh [TT90, App. C]. We show, for example, that every quasi-compact and quasi-separated Deligne–Mumford stack X can be written as an inverse limit of Deligne–Mumford stacks X_λ of finite type over $\operatorname{Spec}\mathbb{Z}$. Such results are sometimes known as "absolute approximation" [TT90, App. C] in contrast with the "standard limit results" [EGA_{IV}, §8] which are relative: given an inverse limit $X = \varprojlim_\lambda X_\lambda$, describe *finitely presented* objects over X in terms of finitely presented objects over X_λ for sufficiently large λ.

We say that an algebraic stack X is of *global type* if étale-locally X is a global quotient stack, cf. Section 2 for a precise definition. Examples of stacks of global type are quasi-compact and quasi-separated schemes, algebraic spaces, Deligne–Mumford stacks and algebraic stacks with quasi-finite (and locally separated) diagonals. For convenience, we also introduce the notion of approximation type. Every stack of global type is of approximation type (Proposition 2.10).

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The main result of this paper, Theorem D, is that any stack of approximation type can be approximated by a noetherian stack. More generally, if $X \to S$ is a morphism between stacks of approximation type, then X is an inverse limit of finitely presented stacks over S.

The primary application of the approximation theorem is the elimination of noetherian, excellency and finiteness hypotheses. When eliminating noetherian hypotheses in statements about finitely presented morphisms $X \to Y$ which are *local* on Y, the basic affine approximation result referred to in the beginning is sufficient, cf. the standard limit results in [EGA_{IV}, §8] and Appendix B. For *global* problems, it is crucial to have Theorem D. Examples of such applications, including generalizations of Chevalley's, Serre's and Zariski's theorems and Chow's lemma, are given in Section 8. Although this paper is written with stacks in mind, most of the applications in §8 are new also when applied to schemes and algebraic spaces. We also answer a question by Grothendieck [EGA_{IV}, Rem. 18.12.9] on integral morphisms affirmatively, cf. Theorem (8.5).

Before stating the main results we need another definition. An algebraic stack X has the *completeness property* if every quasi-coherent sheaf on X is a filtered direct limit of finitely presented sheaves or, equivalently, if the abelian category $\mathbf{QCoh}(X)$ is compactly generated. An algebraic stack X is *pseudo-noetherian* if it is quasi-compact, quasi-separated and X' has the completeness property for every finitely presented morphism $X' \to X$ of algebraic stacks. A key insight during this work was that, for approximation purposes, "pseudo-noetherian", rather than the "completeness property", is the correct notion to work with.

Theorem A (Completeness)**.** *Every stack of approximation type is pseudo-noetherian.*

Every noetherian stack is pseudo-noetherian [LMB00, Prop. 15.4]. In the category of schemes, Theorem A is well-known [EGA_{I}, §6.9] and a slightly weaker result for algebraic spaces is due to Raynaud and Gruson [RG71, Prop. 5.7.8].

Theorem B (Finite coverings)**.** *Let X be a quasi-compact stack with quasi-finite and separated diagonal (resp. a quasi-compact Deligne–Mumford stack with quasi-compact and separated diagonal). Then there exists a scheme Z and a finite, finitely presented and surjective morphism $Z \to X$ that is flat (resp. étale) over a dense quasi-compact open substack $U \subseteq X$.*

When X is a *noetherian Deligne–Mumford stack*, Theorem B is due to G. Laumon and L. Moret-Bailly [LMB00, Thm. 16.6]. When X is of finite type over a noetherian scheme, the existence of a scheme Z and a finite and surjective, but not necessarily generically flat, morphism $Z \to X$ was shown by D. Edidin, B. Hassett, A. Kresch and A. Vistoli [EHKV01, Thm. 2.7].

Before stating the main approximation theorem, we introduce the approximation of various properties. This allows us to unify various approximation results for schemes, algebraic spaces, Deligne–Mumford stacks and so on, in one theorem. Thus, consider the following properties of a morphism of algebraic stacks:

(PA) affine; quasi-affine; representable; separated; locally separated (i.e., diagonal is an immersion); separated diagonal; locally separated diagonal; unramified diagonal; quasi-finite diagonal; affine diagonal; quasi-affine diagonal; finite inertia; abelian inertia; tame inertia (i.e., stabilizer groups are finite and linearly reductive);

(PC) closed immersion; immersion; monomorphism of finite type; unramified; quasi-finite; finite; proper with finite diagonal;

(PI) integral.

These properties are all stable under composition and fppf-local on the target. Also note that affine morphisms have all properties in (PA) and closed immersions have all properties in (PC).

Theorem C (Approximation of properties)**.** *Let S be a quasi-compact algebraic stack and let $\{X_\lambda \to S\}$ be an inverse system of quasi-compact and quasi-separated morphisms of algebraic stacks with affine bonding maps $X_\mu \to X_\lambda$ and limit $X \to S$.*

(i) *Let P be one of the properties in* (PA)*. Then $X \to S$ has property P if and only if there exists an index α such that $X_\lambda \to S$ has property P for every $\lambda \geq \alpha$.*

(ii) *Assume that the morphisms $X_\lambda \to S$ are of finite type and the bonding maps $X_\mu \to X_\lambda$ are closed immersions; hence $X \to S$ is of finite type. Let P be one of the properties in* (PC)*. Then $X \to S$ has property P if and only if there exists an index α such that $X_\lambda \to S$ has property P for every $\lambda \geq \alpha$.*

(iii) *Assume that S is quasi-separated. Then X is a scheme if and only if there exists an index α such that X_λ is a scheme for every $\lambda \geq \alpha$.*

Note that, contrarily to similar results, we do not require that $X_\lambda \to S$ is of finite presentation in Theorem C. This is crucial for Theorem D (iii) and many applications, e.g., Remark (2.2).

Theorem D (Approximation)**.** *Let S be a pseudo-noetherian algebraic stack and let $X \to S$ be a morphism of approximation type (these assumptions are satisfied if X and S are of global type). Then, there exists a* finitely presented *morphism $X_0 \to S$ and an* affine *S-morphism $X \to X_0$. Moreover, $X \to X_0 \to S$ can be chosen such that the following holds.*

(i) *If $X \to S$ is of finite type, then $X \to X_0$ is a closed immersion.*

(ii) *If $X \to S$ has one of the properties in* (PA), (PC) *or* (PI)*, then so has $X_0 \to S$.*

(iii) *If $X \to \operatorname{Spec} \mathbb{Z}$ has one of the properties in* (PA) *then so has $X_0 \to \operatorname{Spec} \mathbb{Z}$. If X is a scheme then so is X_0.*

Furthermore, X can be written as an inverse limit $\varprojlim_\lambda X_\lambda$ of finitely presented S-stacks with affine bonding maps such that for every λ, the factorization $X \to X_\lambda \to S$ satisfies (i)–(iii) *with $X_0 = X_\lambda$. Finally, if $X \to S$ is of finite type (resp. integral), then there is such an inverse system with bonding maps that are closed immersions (resp. finite).*

When X and S are *schemes*, parts of Theorems C and D have been shown by R. W. Thomason and T. Trobaugh [TT90, App. C], B. Conrad [Con07, Thm. 4.3, App. A] and M. Temkin [Tem11, Thm. 1.1.2]. When X and S are *algebraic spaces*, parts of Theorem D were recently obtained independently by B. Conrad, M. Lieblich and M. Olsson [CLO12, §3].

There are also some approximation results for group schemes. If G is a quasi-compact group scheme over a field, then D. Perrin has shown that G is an inverse limit of group schemes of finite type [Per76].

Étale dévissage. The étale dévissage method of [Ryd11b] is the primary technique behind the proofs of Theorems A and D and is prominent in all previous treatments of approximation for algebraic spaces. There is a subtle, yet crucial, difference in our treatment, though. Our dévissage gives statements of the form: given $X' \to X$ surjective and étale, then X can be approximated if X' can be approximated (Proposition 4.11 and Lemma 7.9). This is accomplished by reducing to the case where $X' \to X$ is an étale neighborhood. All other treatments, from Raynaud–Gruson and onwards, would demand that $X' \to X$ is an étale neighborhood such that X' is (quasi-)affine. This approach has recently been formalized as "scallop decompositions" by J. Lurie and can only deal with algebraic spaces. As our inductive approach is not based on quasi-affine neighborhoods, we have to be more careful when formulating the inductive hypothesis, e.g., use "pseudo-noetherian" instead of the "completeness property".

Overview. We begin with some conventions on stacks in Section 1. In Section 2, we define stacks of global type and stacks of approximation type. We show that every quasi-compact algebraic stack with quasi-finite and locally separated diagonal is of global type. In Section 3, we briefly outline the étale dévissage method. In Sections 4 and 5, we prove Theorems A and B. In Section 6, we prove Theorem C *when the morphisms $X_\lambda \to S$ are of finite presentation*. This case of the theorem is essentially independent of Theorems A and B. In Section 7, we prove the general form of Theorems C and D. We conclude with numerous applications of the main theorems in Section 8.

In the appendices, we extend the standard results [$\mathrm{EGA}_{\mathrm{IV}}$, §8–9] on limits and constructible properties from schemes to stacks.

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Contents

1. Stack conventions

We follow the conventions in [LMB00] except that we follow [SP, 026O] and do not require that the diagonal of an algebraic stack is quasi-compact and separated. One reason for this is that stacks with non-separated diagonals naturally appears in the context of [Ryd11b]. On the other hand, very little is lost by assuming that all algebraic stacks are *quasi-separated*, i.e., that the diagonal is quasi-compact and quasi-separated, and most results require this hypothesis. If Y is a quasi-compact and quasi-separated algebraic stack, then $X \to Y$ is quasi-compact if and only if X is so. In particular, an open substack $U \subseteq Y$ is quasi-compact if and only if the morphism $U \to Y$ is quasi-compact.

(1.1) A *presentation* of a stack X is an algebraic space X' and a faithfully flat morphism $X' \to X$ locally of finite presentation. A morphism $f\colon X \to Y$ of stacks is *representable* (resp. *strongly representable*) if $X \times_Y Y'$ is an algebraic space (resp. a scheme) for every scheme Y' and morphism $Y' \to Y$. Note that the property of being representable is fppf-local on the target. Indeed, a morphism is representable if and only if its diagonal is a monomorphism. This is not the case for the property of being strongly representable. A morphism $X \to S$ of stacks is *locally separated* if the diagonal $\Delta_{X/S}$ is an immersion. In particular, every locally separated morphism is representable.

An algebraic stack X is *Deligne–Mumford* if there exists an étale presentation of X, or equivalently, if the diagonal is unramified [LMB00, Thm. 8.1], [SP, 06N3].

An algebraic stack is *noetherian* if it is quasi-compact and quasi-separated and admits a noetherian presentation.

(1.2) *Unramified and étale* — For the definition and general properties of unramified and étale morphisms of stacks, we refer to [Ryd11a, App. B]. In particular, by an unramified morphism we mean a formally unramified morphism that is locally of *finite type* (not necessarily of finite presentation). An étale morphism is a formally étale morphism that is locally of finite presentation. A morphism is unramified if and only if it is locally of finite type and its diagonal is étale. A morphism is étale if and only if it is locally of finite presentation, unramified and flat. We do not require that unramified and étale morphisms are representable.

(1.3) *Quasi-finite* — A morphism $f\colon X \to Y$ of stacks is (locally) *quasi-finite* if f is (locally) of finite type, every fiber of f is discrete and every fiber of the diagonal Δ_f is discrete. Equivalently, f is (locally) quasi-finite if and only if f is (locally) of finite type, every fiber of f is zero-dimensional and every fiber of the diagonal of f is zero-dimensional. Note that the diagonal of a quasi-finite and quasi-separated morphism is quasi-finite.

Given a morphism $f\colon X \to Y$ of algebraic stacks, we say that $h \circ g\colon X \to X_0 \to Y$ is a *factorization* of f if there exists a 2-isomorphism $f \Rightarrow h \circ g$.

The *inertia stack* of a morphism $f\colon X \to Y$ of algebraic stacks is the algebraic stack $I_{X/Y} := X \times_{X \times_Y X} X$. It comes with a representable morphism $I_f\colon I_{X/Y} \to X$ equipped with the structure of a relative group space over X. We say that f has finite (etc.) inertia if I_f is finite (etc.).

By convention, all our inverse systems are filtered and all maps in inverse systems are affine.

2. Stacks of global type and approximation type

In this section, we define stacks of global type and show that every quasi-compact stack with quasi-finite and locally separated diagonal is of global type. We also define stacks of approximation type, which is a natural class of stacks for our purposes. Every stack of global type and every stack of finite presentation over a quasi-compact and quasi-separated scheme or algebraic space is of approximation type.

Definition (2.1). Let X be an algebraic stack. We say that X is

(i) *basic* if $X = [V/\mathrm{GL}_n]$ for some *quasi-affine* scheme V and integer n;
(ii) of *global type* if there exists a representable, étale, finitely presented and surjective morphism $p\colon X' \to X$ such that X' is basic;
(iii) of *s-global type* if there exists a separated, representable, étale, finitely presented and surjective morphism $p\colon X' \to X$ such that X' is basic;
(iv) a *global quotient stack* if $X = [V/\mathrm{GL}_n]$ where V is an algebraic space.

Remark **(2.2).** *Relation with the resolution property* — B. Totaro has shown that a normal noetherian stack is a basic stack if and only if it has the resolution property [Tot04]. By recent work of P. Gross, this also holds for non-normal noetherian stacks [Gro10, Thm. 6.3.1]. Using Theorem C, one can give a very satisfactory proof of this result that is also valid without noetherian hypotheses [Gro13]. Thus, a stack X is of global type if and only if the resolution property holds étale-locally on X.

Every basic stack has affine diagonal. There are very few examples of stacks with affine diagonal that are known to be non-basic. For example, there is not a single example of a non-basic separated scheme or Deligne–Mumford stack [Tot04, Question 1] although S. Payne has given some evidence that a certain proper toric three-fold is not basic [Pay09].

The following example shows that many global quotient stacks are of global type.

Example (2.3). Let G be an affine smooth group scheme over $\operatorname{Spec}\mathbb{Z}$ with connected fibers, e.g., $G = \mathrm{GL}_{n,\mathbb{Z}}$. If X is a *normal noetherian scheme* with an action of G, then $[X/G]$ is of s-global type. Indeed, if X is quasi-projective, then $[X/G]$ has the resolution property by [Tot04, Thm. 2.1 (2)] and a result of Sumihiro [Sum75, Thm. 3.8] states that there exists a Zariski covering $U \to [X/G]$ such that $X \times_{[X/G]} U$ is quasi-projective.

Question **(2.4).** Is every noetherian global quotient stack of s-global type?

Remark **(2.5).** Note that being of global type is not a purely local condition since $p\colon X' \to X$ is required to be of finite presentation, so X is quasi-compact and quasi-separated. We require that p is representable as currently there is no suitable dévissage for non-representable étale morphisms.

If X is a stack of global type, then X is quasi-compact, quasi-separated and Δ_X is locally separated with affine fibers, cf. [Ryd11b, App. A]. If X is a stack of s-global type, then Δ_X is also quasi-affine. Note that there exist stacks of global type with non-separated diagonals, e.g., every quasi-compact and quasi-separated Deligne–Mumford stack is of global type.

I am not aware of any example of a stack with quasi-affine diagonal that is not of global type. There are, on the other hand, examples of stacks with quasi-affine diagonal that are not global quotient stacks. One such example, with affine diagonal, is the $\mathbb{G}_m$-gerbe over a complex surface Y corresponding to a non-torsion element of the cohomological Brauer group $H^2_{\text{ét}}(Y, \mathbb{G}_m)$ [EHKV01, Ex. 3.12]. Another example is the two-dimensional Deligne–Mumford stack of [EHKV01, Ex. 2.21] which has quasi-affine diagonal. These two examples are easily seen to be of global type. A third example of a stack of global type that is not a global quotient stack, albeit not with quasi-affine diagonal, is the stack $\mathfrak{M}_0^{\leq m}$ of prestable curves of genus 0 with at most m nodes, for $m \geq 2$ [Kre13, §5].

Proposition (2.6). *Let X be an algebraic stack with a finite flat presentation $p\colon V \to X$ such that V is quasi-affine. Then X is a basic stack.*

Proof. Let $\mathcal{L} = p_*\mathcal{O}_V$ which is a locally free sheaf of finite rank. Let $x\colon \operatorname{Spec} k \to X$ be a point and let G_x be the stabilizer group scheme of x. The stabilizer group scheme acts on the k-vector space $\mathcal{L}_x$ and this coaction is faithful since the stabilizer action on the subscheme $V_x \hookrightarrow \mathbf{V}(\mathcal{L}_x)$ is free. Replacing $\mathcal{L}$ with the direct sum of $\mathcal{L}$ and a free sheaf, we can further assume that $\mathcal{L}$ is of constant rank r.

Let $Z = \underline{Isom}_X(\mathcal{O}_X^r, \mathcal{L}) \subseteq \mathbf{V}\big((\mathcal{L}^\vee)^r\big)$ be the frame bundle of $\mathcal{L}$. The morphism $Z \to X$ is a $\mathrm{GL}_{r,\mathbb{Z}}$-torsor and Z is an algebraic space since the action of G_x on the fiber Z_x is free. Since $Z \to X$ is affine, we have that $Z \times_X V$ is quasi-affine. As $Z \times_X V \to Z$ is a finite flat presentation, we conclude that Z is quasi-affine as well by [Ryd11b, Lem. C.1]. Thus $X = [Z/\mathrm{GL}_r]$ is a basic stack. □

Corollary (2.7). *Every quasi-compact algebraic stack with quasi-finite and locally separated (resp. separated) diagonal is of global type (resp. s-global type).*

Proof. Let X be a stack with quasi-finite and locally separated (resp. separated) diagonal. By [Ryd11b, Thm. 7.2], there exists a representable (resp. representable and separated) étale surjective morphism $X' \to X$ of finite presentation and a finite flat presentation $V \to X'$ with V quasi-affine.

Since X' is a basic stack (Proposition 2.6), we have, by definition, that X is of global type (resp. s-global type). □

The following result, which partly generalizes the two previous results but depends on [Gro13], is not used in this paper but included for completeness. In particular, it justifies the usage of étale in the definition of s-global type.

Proposition (2.8). *Let $f\colon X \to Y$ be a morphism of algebraic stacks.*

(i) *Assume that f is quasi-affine. If Y is basic (resp. of s-global type, resp. of global type), then so is X.*
(ii) *Assume that f is finite and faithfully flat of finite presentation. If X is basic, then so is Y.*
(iii) *Assume that f is quasi-finite, representable, separated and faithfully flat of finite presentation. If X is of s-global type, then so is Y.*

In particular, in the definition of s-global type, we can replace "étale" with "quasi-finite and flat".

Proof. (i) is trivial from the definitions. (ii) follows from [Gro13, Prop. 4.3 (vii)] taking into account [Gro13, Cor. 5.9]. For (iii), we can assume that X is basic. Then using [Ryd11b, Thm. 6.3 (i)] and (i), we can assume that f is finite and the result follows from (ii). □

Definition (2.9). We say that a morphism $f\colon X \to Y$ of algebraic stacks is of *strict approximation type* if f can be written as a composition of affine morphisms and finitely presented morphisms. We say that f is of *approximation type* if there exists a surjective representable and finitely presented étale morphism $p\colon X' \to X$ such that $f \circ p$ is of strict approximation type. We say that an algebraic stack X is of (strict) approximation type if $X \to \operatorname{Spec} \mathbb{Z}$ is of (strict) approximation type.

We begin with the usual sorites of morphisms of (strict) approximation type.

Proposition (2.10). —

(i) *Finitely presented morphisms and quasi-affine morphisms are of strict approximation type.*
(ii) *Every morphism of approximation type is quasi-compact and quasi-separated.*
(iii) *Basic stacks are of strict approximation type. Stacks of global type, e.g., quasi-compact and quasi-separated Deligne–Mumford stacks, are of approximation type.*
(iv) *If $f\colon X \to Y$ is of (strict) approximation type and $Y' \to Y$ is a morphism, then $f'\colon X \times_Y Y' \to Y'$ is of (strict) approximation type.*
(v) *If $f\colon X \to Y$ and $g\colon Y \to Z$ are of (strict) approximation type then so is $g \circ f$.*
(vi) *If $f_1\colon X_1 \to Y_1$ and $f_2\colon X_2 \to Y_2$ are of (strict) approximation type then so is $f_1 \times f_2$.*
(vii) *If $f\colon X \to Y$ and $g\colon Y \to Z$ are morphisms such that $g \circ f$ and Δ_g are of (strict) approximation type, then so is f.*

(viii) *If $f\colon X \to Y$ is of (strict) approximation type, then so is $\Delta_f\colon X \to X \times_Y X$.*

In particular, morphisms between stacks of global type are of approximation type.

Proof. (i), (ii), (iv), (v) and (vi) are obvious and (vii) follows from a standard argument.

(iii) Let $X = [V/\mathrm{GL}_n]$ with V quasi-affine. Then there is an induced quasi-affine morphism $X \to \mathrm{BGL}_{n,\mathbb{Z}}$, so X is of strict approximation type.

(viii) If $f = f_n \circ f_{n-1} \circ \dots \circ f_1$ is a composition of affine morphisms and finitely presented morphisms, then so is Δ_f since it is a composition of pull-backs of the Δ_{f_i}'s. If $p\colon X' \to X$ is a surjective representable and finitely presented étale morphism such that $f \circ p$ is of strict approximation type, then $\Delta_f \circ p = (p \times p) \circ \Delta_{f\circ p}$ is of strict approximation type. $\square$

Proposition (2.11). *Let $f\colon X \to Y$ be a morphism of algebraic stacks.*

(i) *Let $g\colon Y' \to Y$ be surjective, representable and étale of finite presentation. If $f'\colon X \times_Y Y' \to Y'$ is of approximation type, then so is f.*

(ii) *Let $p\colon X' \to X$ be surjective, representable and étale of finite presentation. If $f \circ p$ is of approximation type, then so is f.*

Proof. Immediate from the definition of approximation type. $\square$

We will now give an analogue of Proposition (2.11) for finite flat morphisms which also is valid for strict approximation type.

Proposition (2.12). *Let $f\colon X \to Y$ be a morphism of algebraic stacks.*

(i) *Let $g\colon Y' \to Y$ be finite and faithfully flat of finite presentation. If $f'\colon X \times_Y Y' \to Y'$ is of (strict) approximation type, then so is f.*

(ii) *Let $p\colon X' \to X$ be finite and faithfully flat of finite presentation and assume that X is quasi-compact. If $f \circ p$ is of (strict) approximation type, then so is f.*

Proof. (i) We will make use of the Weil restriction of stacks along g [Ryd11c, §3]. Recall that the Weil restriction is a 2-functor $\mathbf{R}_g\colon \mathbf{Stack}_{/Y'} \to \mathbf{Stack}_{/Y}$ from stacks over Y' to stacks over Y. If Z is an algebraic stack over Y', then so is $\mathbf{R}_g(Z) \to Y$. If $h\colon Z_1 \to Z_2$ is a morphism of algebraic stacks over Y', then there is an induced morphism $\mathbf{R}_g(h)\colon \mathbf{R}_g(Z_1) \to \mathbf{R}_g(Z_2)$. If h is affine (resp. of finite presentation, resp. étale and surjective, resp. representable, resp. a monomorphism) then so is $\mathbf{R}_g(h)$. In particular, if f' is of (strict) approximation type then so is $\mathbf{R}_g(f')\colon \mathbf{R}_g(X \times_Y Y') \to \mathbf{R}_g(Y') = Y$. As $\mathbf{R}_g$ is the right adjoint to the pull-back functor $g^{-1}\colon \mathbf{Stack}_{/Y} \to \mathbf{Stack}_{/Y'}$ we have unit and counit maps

$$\eta\colon X \to \mathbf{R}_g(X \times_Y Y')$$
$$\epsilon\colon \mathbf{R}_g(X \times_Y Y') \times_Y Y' \to X \times_Y Y'$$

such that

$$f = \mathbf{R}_g(f') \circ \eta$$
$$\mathbf{R}_g(f') \times_Y \mathrm{id}_{Y'} = f' \circ \epsilon$$
$$\mathrm{id}_{X \times_Y Y'} = \epsilon \circ (\eta \times_Y \mathrm{id}_{Y'}).$$

It is thus enough to show that η is of (strict) approximation type. Since $\mathbf{R}_g(f')$ and f' are of (strict) approximation type, so are ϵ and $\eta\times_Y \mathrm{id}_{Y'}$. We further observe that if f is arbitrary (resp. representable, resp. a monomorphism) then so is ϵ and it follows that η is representable (resp. a monomorphism, resp. an isomorphism). We can thus replace f with η and $g\colon Y'\to Y$ with its pull-back $\mathbf{R}_g(X\times_Y Y')\times_Y Y'\to \mathbf{R}_g(X\times_Y Y')$ and further assume that f is representable (resp. a monomorphism, resp. an isomorphism). Repeating the argument twice settles (i).

(ii) We will make use of the *Hilbert stack of points* [Ryd11c]. Since X is quasi-compact, and the fiber rank of p is locally constant, we can assume that p has constant rank d. This induces a morphism $X\to \mathscr{H}^d_{Y/Y}$ and a cartesian diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Z \arrow[d] & \\ X \arrow[r] & \mathscr{H}^d_{Y/Y} \arrow[r] & Y \end{tikzcd}$$

where $Z\to \mathscr{H}^d_{Y/Y}$ is the universal finite flat family of constant rank d. Since $\mathscr{H}^d_{Y/Y}\to Y$ is of finite presentation, we have that $X'\to Z$ is of (strict) approximation type, so $X\to Y$ is of (strict) approximation type by (i). □

For our immediate purposes, we only need Proposition (2.12) for finite *étale* coverings. The proof can then be simplified by replacing the Weil restriction and the Hilbert stack with symmetric products and $\mathrm{B}\mathfrak{S}_d\times Y$. However, Proposition (2.12) together with [Ryd11b, Thm. 6.3 (ii)] shows that in the definition of approximation type, we can essentially replace étale with quasi-finite flat. To be precise, if $f\colon X\to Y$ is a morphism of algebraic stacks such that X is quasi-compact and quasi-separated, then f is of approximation type if and only if there exists a representable, locally separated, quasi-finite and faithfully flat morphism $p\colon X'\to X$ of finite presentation such that $f\circ p$ is of strict approximation type.

It should be noted that the only reason for insisting upon p being representable in Definitions (2.1) and (2.9) is that currently we only have a nice étale dévissage for representable morphisms. This is also the reason behind the local separatedness assumption in Corollary (2.7).

Questions **(2.13).** Is the notion of (strict) approximation type fppf-local on the target? Is every noetherian stack of approximation type? Is every quasi-compact and quasi-separated stack of approximation type? Is every representable morphism of approximation type? Is every quasi-compact stack with quasi-affine diagonal of global type?

3. Étale dévissage

The étale dévissage method reduces questions about general étale morphisms to étale morphisms of two basic types. The first type is finite étale coverings. The second is *étale neighborhoods* or equivalently pushouts of étale morphisms and open immersions — the étale analogue of open coverings consisting of two open subsets. For the reader's convenience, we summarize the main results of [Ryd11b].

Definition (3.1). Let X be an algebraic stack and let $Z \hookrightarrow |X|$ be a closed subset. An étale morphism $p\colon X' \to X$ is an *étale neighborhood of* Z if $p|_{Z_{\mathrm{red}}}$ is an isomorphism.

Theorem (3.2) ([Ryd11b, Thm. A])**.** *Let X be an algebraic stack and let $U \subseteq X$ be an open substack. Let $f\colon X' \to X$ be an étale neighborhood of $X \setminus U$ and let $U' = f^{-1}(U)$. The natural functor*

$$(|_U, f^*)\colon \mathbf{QCoh}(X) \to \mathbf{QCoh}(U) \times_{\mathbf{QCoh}(U')} \mathbf{QCoh}(X')$$

is an equivalence of categories.

Theorem (3.3) ([Ryd11b, Thm. B])**.** *Let X be an algebraic stack and let $j\colon U \to X$ be an open immersion. Let $p\colon X' \to X$ be an étale neighborhood of $X \setminus U$ and let $j'\colon U' \to X'$ be the pull-back of j. Then X is the pushout in the category of algebraic stacks of $p|_U$ and j'.*

Theorem (3.4) ([Ryd11b, Thm. C])**.** *Let X' be a quasi-compact and quasi-separated algebraic stack, let $j'\colon U' \to X'$ be a quasi-compact open immersion and let $p_U\colon U' \to U$ be a finitely presented étale morphism. Then, the pushout X of j' and p_U exists in the category of quasi-compact and quasi-separated algebraic stacks. The resulting co-cartesian diagram*

$$\begin{tikzcd} U' \arrow[r, "j'"] \arrow[d, "p_U"'] \arrow[dr, phantom, "\square"] & X' \arrow[d, "p"] \\ U \arrow[r, "j"'] & X \end{tikzcd}$$

is also cartesian, j is a quasi-compact open immersion and p is an étale and finitely presented neighborhood of $X \setminus U$.

Theorem (3.5) ([Ryd11b, Thm. D])**.** *Let X be a quasi-compact and quasi-separated algebraic stack and let $\mathbf{E}$ be the 2-category of finitely presented étale morphisms $Y \to X$. Let $\mathbf{D} \subseteq \mathbf{E}$ be a full subcategory such that*

(D1) *if $Y \in \mathbf{D}$ and $(Y' \to Y) \in \mathbf{E}$, then $Y' \in \mathbf{D}$,*
(D2) *if $Y' \in \mathbf{D}$ and $Y' \to Y$ is finite, surjective and étale, then $Y \in \mathbf{D}$, and*
(D3) *if $j\colon U \to Y$ and $f\colon Y' \to Y$ are morphisms in $\mathbf{E}$ such that j is an open immersion, f is an étale neighborhood of $Y \setminus U$ and $U, Y' \in \mathbf{D}$, then $Y \in \mathbf{D}$.*

Then, if $(Y' \to Y) \in \mathbf{E}$ is representable *and surjective and $Y' \in \mathbf{D}$, it follows that $Y \in \mathbf{D}$. In particular, if there exists a* representable *and surjective morphism $Y \to X$ in $\mathbf{E}$ with $Y \in \mathbf{D}$, then $\mathbf{D} = \mathbf{E}$.*

Note that the morphisms in $\mathbf{E}$ are not necessarily representable nor separated. In Theorem (3.4), even if X' and U have separated diagonals, the pushout X need not unless p_U is representable. We are thus naturally led to include algebraic stacks with non-separated diagonals.

4. Approximation of modules and algebras

In this section, we prove Theorem A, that is, that every stack of approximation type is *pseudo-noetherian*. It is known that noetherian stacks are pseudo-noetherian [LMB00, Prop. 15.4] but the non-noetherian case requires completely different methods. First we prove that stacks of strict approximation type are pseudo-noetherian and then we deduce the theorem for stacks of approximation type by étale dévissage.

(4.1) Let X be a quasi-compact and quasi-separated algebraic stack. Let $\mathbf{C}$ be one of the following categories.

(i) The category of quasi-coherent $\mathcal{O}_X$-modules.
(ii) The category of quasi-coherent $\mathcal{O}_X$-algebras.
(iii) The category of integral quasi-coherent $\mathcal{O}_X$-algebras.

Here integral has the meaning as in "integral closure" not as in "integral domain". If $U \subseteq X$ is an open substack we denote the corresponding category of $\mathcal{O}_U$-modules by $\mathbf{C}_U$. Consider the following statements.

Completeness

(C1) Every object in $\mathbf{C}$ is the direct limit of its subobjects of finite type.
(C2) Every object in $\mathbf{C}$ is a filtered direct limit of finitely presented objects in $\mathbf{C}$.

Presentation — Let $\mathcal{F}$ be an object in $\mathbf{C}$ of finite type.

(P1) There exists a finitely presented object $\mathcal{P}$ and a surjection $\mathcal{P} \twoheadrightarrow \mathcal{F}$.
(P2) There is a filtered direct system of finitely presented objects in $\mathbf{C}$ with surjective bonding maps and limit $\mathcal{F}$.

Extension — Let $U \subseteq X$ be a quasi-compact open substack.

(E1) If $\mathcal{F}_U \in \mathbf{C}_U$ is of finite type (resp. finite presentation), then there exists an object $\mathcal{F} \in \mathbf{C}$ of finite type (resp. finite presentation) such that $\mathcal{F}|_U = \mathcal{F}_U$.
(E2) If $\mathcal{G} \in \mathbf{C}$ is arbitrary and $\mathcal{F}_U \in \mathbf{C}_U$ is of finite type (resp. finite presentation), together with a homomorphism $u\colon \mathcal{F}_U \to \mathcal{G}|_U$, then there exists an object $\mathcal{F} \in \mathbf{C}$ of finite type (resp. finite presentation) and a homomorphism $v\colon \mathcal{F} \to \mathcal{G}$ extending $\mathcal{F}_U$ and u. To be precise, there exists an isomorphism $\theta\colon \mathcal{F}|_U \to \mathcal{F}_U$ such that $v|_U = u \circ \theta$.

Note that (C1) follows from (C2), that (P1) follows from (P2) and that (E1) is a special case of (E2) (take $\mathcal{G} = 0$). Also, given $\mathcal{F}_U$, $\mathcal{G}$ and u as in (E2), there is a *universal* extension $v\colon \mathcal{F} \to \mathcal{G}$ of u if we drop the condition that $\mathcal{F}$ is of finite type. Indeed, if $j\colon U \to X$ is the inclusion morphism, then the universal solution is $\mathcal{F} = \mathcal{G} \times_{j_*j^*\mathcal{G}} j_*\mathcal{F}_U$ together with the projection onto the first factor. If $\mathbf{C}$ is the category of integral $\mathcal{O}_X$-algebras, then the universal solution is the integral closure of $\mathcal{O}_X$ in $\mathcal{F}$ (as a subring of $\mathcal{F}$). If u is injective then so is v.

Definition (4.2). Let X be a quasi-compact and quasi-separated stack. We say that X has the *completeness property* if the six properties (C1), (C2), (P1), (P2), (E1) and (E2) hold for X and the categories of quasi-coherent $\mathcal{O}_X$-modules, $\mathcal{O}_X$-algebras and integral $\mathcal{O}_X$-algebras.

In the introduction, the completeness property only entailed (C2) for the category of quasi-coherent $\mathcal{O}_X$-modules but, as we will see in Lemma (4.3), the two definitions are equivalent. Note that if X has the completeness property, then so has $U \subseteq X$ for any quasi-compact open substack. We also introduce the following auxiliary condition.

(C2*) For every object $\mathcal{F} \in \mathbf{C}$, there is a filtered direct system of finitely presented objects $\mathcal{F}_\lambda \in \mathbf{C}$ and a surjection $\varinjlim_\lambda \mathcal{F}_\lambda \twoheadrightarrow \mathcal{F}$.

Lemma (4.3). *Let X be a quasi-compact and quasi-separated stack. Let $\mathbf{C}$ be one of the three categories in* (4.1)*. Then the following conditions are equivalent.*

(i) (C2) *holds for X and $\mathbf{C}$.*
(ii) (C2*) *holds for X and $\mathbf{C}$.*
(iii) (C1) *and* (P1) *hold for X and $\mathbf{C}$.*
(iv) *X has all six properties for $\mathbf{C}$.*

Moreover, if X has property (C2) *for the category of quasi-coherent modules, then X has the completeness property.*

Proof. Clearly (C2) $\Longrightarrow$ (C2*) $\Longrightarrow$ (C1)+(P1) (to see the second implication, pass to a presentation of X by an affine scheme). As we noted above, (E1) is a special case of (E2). We will show three other implications from which the first part of the lemma follows.

(C1)+(P1) $\Longrightarrow$ (P2): Let $\mathcal{F}$ be of finite type and let $\mathcal{P} \twoheadrightarrow \mathcal{F}$ be a surjection with $\mathcal{P}$ finitely presented. Let $\mathcal{K} \subseteq \mathcal{P}$ be the kernel. Then $\mathcal{K}$ is the limit of its submodules (or subideals) $\mathcal{K}_\lambda$ of finite type and it follows that $\mathcal{F} = \varinjlim_\lambda \mathcal{P}/\mathcal{K}_\lambda$ is a limit of finitely presented objects.

(C1)+(P1) $\Longrightarrow$ (C2): Let $\mathcal{F}$ be a quasi-coherent sheaf. Then $\mathcal{F} = \varinjlim_{\lambda \in L} \mathcal{F}_\lambda$ where $\mathcal{F}_\lambda$ is of finite type. Let $\mathcal{P}_\lambda$ be a finitely presented object with a surjection onto $\mathcal{F}_\lambda$. For a finite subset $J \subseteq L$ we let $\mathcal{P}_J$ be the coproduct of $\{\mathcal{P}_\lambda\}_{\lambda \in J}$ in $\mathbf{C}$ and let $\mathcal{K}_J$ be the kernel of the induced homomorphism $\mathcal{P}_J \to \mathcal{F}$. Consider the set of pairs $\alpha = (J, \mathcal{R}_J)$ where $J \subseteq L$ is a finite subset and $\mathcal{R}_J \subseteq \mathcal{K}_J$ is a submodule (or subideal) of finite type. Then $\mathcal{F} = \varinjlim_\alpha \mathcal{P}_J/\mathcal{R}_J$ is a filtered direct limit of finitely presented objects (cf. proof of [$\mathrm{EGA_I}$, Cor. 6.9.12]).

(C1)+(P1) $\Longrightarrow$ (E2): Let $\mathcal{F}_U$ be a quasi-coherent sheaf on U of finite type (resp. of finite presentation) and let $u\colon \mathcal{F}_U \to \mathcal{G}|_U$ be a homomorphism as in (E2). Let $v\colon \mathcal{F} \to \mathcal{G}$ be the universal extension. Then as $\mathcal{F}_U = \mathcal{F}|_U$ is of finite type, it follows from (C1) that there exists a subsheaf $\mathcal{F}' \subseteq \mathcal{F}$ of finite type which restricts to $\mathcal{F}_U$. If $\mathcal{F}_U$ is finitely presented, write $\mathcal{F}' = \mathcal{P}/\mathcal{K}$ with $\mathcal{P}$ of finite presentation. Then $\mathcal{K}|_U$ is of finite type and hence by (C1) there exists a submodule (or subideal) $\mathcal{K}' \subseteq \mathcal{K}$ of finite type which restricts to $\mathcal{K}|_U$. The homomorphism $\mathcal{P}/\mathcal{K}' \twoheadrightarrow \mathcal{F}' \hookrightarrow \mathcal{F} \to \mathcal{G}$ is the requested extension of u.

To prove the last statement, assume that X has property (C2) for the category of quasi-coherent sheaves of modules. Let $\mathcal{A}$ be a sheaf of algebras on X. Considering $\mathcal{A}$ as an $\mathcal{O}_X$-module, we can then write $\mathcal{A} = \varinjlim_\lambda \mathcal{F}_\lambda$ as a filtered direct limit of finitely presented modules. If we then let $\mathcal{A}_\lambda$ be the symmetric algebra of $\mathcal{F}_\lambda$, we have a surjection $\varinjlim_\lambda \mathcal{A}_\lambda \twoheadrightarrow \mathcal{A}$ as in (C2*). This settles the completeness property for the category of algebras.

If $\mathcal{A}$ is an integral algebra, then it is a direct limit of its integral subalgebras since any subalgebra of an integral algebra is integral. This settles (C1) for the category of integral algebras. If $\mathcal{A}$ is of finite type then we can, using (P2) for the category of algebras, write $\mathcal{A}$ as a filtered direct limit of finitely presented algebras $\mathcal{B}_\lambda$ with surjective bonding maps. Then $\mathcal{B}_\lambda$ is integral for sufficiently large λ. Indeed, this is easily verified after passing to an affine presentation. This shows (P1) for the category of integral algebras. $\square$

Remark **(4.4).** Let X be a stack with the completeness property and let $\mathcal{F}$ be a sheaf in one of the categories referred to above. If U is a quasi-compact open substack such that $\mathcal{F}|_U$ is of finite type (resp. of finite presentation), then $\mathcal{F}$ is the direct limit of its finite type subsheaves (resp. a filtered direct limit of finitely presented sheaves) $\mathcal{F}_\lambda$ such that $\mathcal{F}_\lambda|_U \to \mathcal{F}|_U$ is an isomorphism. Indeed, this follows by a similar argument as in the proof that (C1)+(P1) implies (C2) above.

Remark **(4.5).** *Generators* — Recall that a subset $\mathbf{G} \subseteq \mathbf{C}$ is *generating* if a morphism $f\colon \mathcal{F} \to \mathcal{G}$ in $\mathbf{C}$ is zero if and only if $f \circ p = 0$ for every morphism $p\colon \mathcal{P} \to \mathcal{F}$ with $\mathcal{P} \in \mathbf{G}$. We introduce the following conditions for the categories in (4.1).

(G1) The objects of finite type generate $\mathbf{C}$.
(G2) The objects of finite presentation generate $\mathbf{C}$.

It is straight-forward to deduce that (C1) $\iff$ (G1) and that (C2) $\implies$ (G2) $\implies$ (G1)+(P1) so (G2) for the category of quasi-coherent modules is equivalent to the completeness property. Moreover, the compact objects in the categories of (4.1) are exactly the finitely presented objects. Thus, condition (G2), or equivalently (C2), holds for $\mathbf{C}$ if and only if $\mathbf{C}$ is compactly generated.

Proposition (4.6). *Let X be a quasi-compact and quasi-separated stack with the completeness property and let $f\colon X' \to X$ be quasi-affine. Then X' has the completeness property.*

Proof. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_{X'}$-module. Since X has the completeness property, we can write $f_*\mathcal{F} = \varinjlim_\lambda \mathcal{G}_\lambda$ as a filtered direct limit of finitely presented $\mathcal{O}_X$-modules. As f is quasi-affine, the counit homomorphism $f^*f_*\mathcal{F} \to \mathcal{F}$ is surjective. We thus obtain a surjection $\varinjlim_\lambda f^*\mathcal{G}_\lambda \twoheadrightarrow \mathcal{F}$ so condition (C2*) holds for X' and the stack X' has the completeness property by Lemma (4.3). $\square$

For an affine scheme properties (C1) and (P1) are straight-forward and hence any quasi-affine scheme has the completeness property. More generally, the completeness property has been shown for quasi-compact and quasi-separated schemes by Grothendieck [EGA$_{\mathrm{I}}$, §6.9], for *noetherian* algebraic spaces by Knutson [Knu71, Thm. III.1.1, Cor. III.1.2] and for *noetherian* algebraic stacks by Laumon and Moret-Bailly [LMB00, Prop. 15.4].

Definition (4.7). An algebraic stack X is *pseudo-noetherian* if it is quasi-compact, quasi-separated and X' has the completeness property for any finitely presented morphism $X' \to X$ of algebraic stacks.

In particular, any noetherian stack is pseudo-noetherian.

Proposition (4.8). *Let S be a pseudo-noetherian stack and let $X \to S$ be of strict approximation type.*

(i) *There is a factorization of $X \to S$ into an affine morphism $X \to X_0$ followed by a finitely presented morphism $X_0 \to S$.*
(ii) *X is pseudo-noetherian.*

In particular, stacks of strict approximation type (e.g., quasi-affine schemes) are pseudo-noetherian.

Proof. It is enough to prove (i) when there is a factorization $X \to Y \to S$ such that $X \to Y$ is finitely presented and $Y \to S$ is affine. Since S has the completeness property, we can write $Y = \varprojlim_\lambda Y_\lambda$ where $Y_\lambda \to S$ are finitely presented and affine morphisms. For sufficiently large λ, there is a morphism $X_\lambda \to Y_\lambda$ of finite presentation between algebraic stacks such that $X = X_\lambda \times_{Y_\lambda} Y$. This follows from Proposition (B.2). The requested factorization is obtained by letting $X_0 = X_\lambda$ since $X \to X_\lambda$ is affine and $X_\lambda \to Y_\lambda \to S$ is of finite presentation.

(ii) Let $X' \to X$ be of finite presentation. We have to show that X' has the completeness property. As $X' \to X \to S$ is of strict approximation type we have a factorization $X' \to X_0' \to S$ consisting of an affine morphism followed by a finitely presented morphism. It follows from Proposition (4.6) that X' has the completeness property since X_0' has the completeness property by definition.

The last statement follows from the fact that $\operatorname{Spec}\mathbb{Z}$ is pseudo-noetherian. □

The proof of the following result is inspired by a similar argument due to P. Gross.

Lemma (4.9). *Let X be a quasi-compact and quasi-separated stack and let $p\colon X' \to X$ be finite and faithfully flat of finite presentation. If X' has the completeness property, then so has X. Thus, X is pseudo-noetherian if and only if X' is pseudo-noetherian.*

Proof. Assume that X' has the completeness property. By Lemma (4.3) it is enough to show that (C2*) holds for X. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Recall that $p_*\colon \mathbf{QCoh}(X') \to \mathbf{QCoh}(X)$ has a right adjoint $p^!\colon \mathbf{QCoh}(X) \to \mathbf{QCoh}(X')$ defined by

$$p^!\mathcal{F} = p^{-1}\mathcal{H}om_{\mathcal{O}_X}(p_*\mathcal{O}_{X'}, \mathcal{F}) \otimes_{p^{-1}p_*\mathcal{O}_{X'}} \mathcal{O}_{X'}.$$

Moreover, the counit homomorphism $p_*p^!\mathcal{F} \to \mathcal{F}$ is surjective since p is faithfully flat. Write $p^!\mathcal{F}$ as a filtered direct limit of finitely presented $\mathcal{O}_{X'}$-modules $\mathcal{G}_\lambda$. Then $\varinjlim_\lambda p_*\mathcal{G}_\lambda = p_*p^!\mathcal{F} \to \mathcal{F}$ is surjective and (C2*) holds for X. □

For our immediate purposes we only need Lemma (4.9) for finite étale coverings in which case $p^! = p^{-1}$.

We now come to the step that involves étale neighborhoods. The method is inspired by Raynaud and Gruson's proof of property (C1) for quasi-compact and quasi-separated algebraic spaces [RG71, Prop. 5.7.8].

Lemma (4.10). *Let X be a quasi-compact and quasi-separated stack, let $U \subseteq X$ be a quasi-compact open substack and let $p\colon X' \to X$ be a finitely presented étale neighborhood of $X \setminus U$. If X' and U have the completeness property, then so has X. In particular, if X' and U are pseudo-noetherian, then so is X.*

Proof. Let $U' = p^{-1}(U)$. By Lemma (4.3) it is enough to show that (C1) and (P1) hold for X.

We begin with (C1). Let $\mathcal{F}$ be a quasi-coherent sheaf on X. As X' has property (C1), we have that $p^*\mathcal{F}$ is the direct limit of its subsheaves of finite type. It is thus enough to show that if $\mathcal{G}' \subseteq p^*\mathcal{F}$ is of finite type, then there exists $\mathcal{H} \subseteq \mathcal{F}$ of finite type such that $\mathcal{G}' \subseteq p^*\mathcal{H}$. As U has property (C1), there is a subsheaf $\mathcal{H}_U \subseteq \mathcal{F}|_U$ of finite type on U such that $\mathcal{G}'|_{U'} \subseteq (p|_U)^*\mathcal{H}_U$. Let $\overline{\mathcal{G}'} \subseteq p^*\mathcal{F}$ be the universal extension of $(p|_U)^*\mathcal{H}_U \subseteq p^*\mathcal{F}|_{U'}$; then $\mathcal{G}' \subseteq \overline{\mathcal{G}'} \subseteq p^*\mathcal{F}$. As $\mathcal{G}'$ and $\overline{\mathcal{G}'}|_{U'} = (p|_U)^*\mathcal{H}_U$ are of finite type, it follows from property (C1) on X' that there exists a subsheaf $\mathcal{H}' \subseteq \overline{\mathcal{G}'}$ of finite type, containing $\mathcal{G}'$ and restricting to $(p|_U)^*\mathcal{H}_U$ over U'. By Theorem (3.2), there is a subsheaf $\mathcal{H} \subseteq \mathcal{F}$ of finite type with isomorphisms $\mathcal{H}|_U \cong \mathcal{H}_U$ and $p^*\mathcal{H} \cong \mathcal{H}'$. This settles property (C1).

We continue with property (P1). Let $\mathcal{F}$ be a quasi-coherent sheaf on X of finite type. As U has property (P2), we can write $\mathcal{F}|_U$ as a direct limit $\varinjlim \mathcal{P}_{U,\lambda}$ of finitely presented sheaves on U with surjective bonding maps. As X' has property (P1), there is a finitely presented sheaf $\mathcal{Q}'$ on X' and a surjection $\mathcal{Q}' \twoheadrightarrow p^*\mathcal{F}$. For sufficiently large λ we have a factorization

$$\mathcal{Q}'|_{U'} \to (p|_U)^*\mathcal{P}_{U,\lambda} \twoheadrightarrow p^*\mathcal{F}|_{U'},$$

cf. proof of [$\mathrm{EGA_{IV}}$, Thm. 8.5.2]. Moreover, after increasing λ we may assume that the homomorphism $\mathcal{Q}'|_{U'} \to (p|_U)^*\mathcal{P}_{U,\lambda}$ is surjective.

Let $\mathcal{K}' = \ker(\mathcal{Q}' \twoheadrightarrow p^*\mathcal{F})$ and $\mathcal{N}'_{U'} = \ker(\mathcal{Q}'|_{U'} \twoheadrightarrow (p|_U)^*\mathcal{P}_{U,\lambda}) \subseteq \mathcal{K}'|_{U'}$. As $\mathcal{N}'_{U'}$ is of finite type, there exists, by (E2), a subsheaf $\mathcal{N}' \subseteq \mathcal{K}'$ of finite type such that $\mathcal{N}'|_{U'} = \mathcal{N}'_{U'}$. Let $\mathcal{P}' = \mathcal{Q}'/\mathcal{N}'$. This is a finitely presented sheaf on X' with a surjection onto $p^*\mathcal{F}$ such that $\mathcal{P}'|_{U'} = (p|_U)^*\mathcal{P}_{U,\lambda}$. By Theorem (3.2), there is a finitely presented $\mathcal{O}_X$-module $\mathcal{P}$ and a surjection $\mathcal{P} \twoheadrightarrow \mathcal{F}$ which restricts to $\mathcal{P}_{U,\lambda} \twoheadrightarrow \mathcal{F}|_U$ and $\mathcal{P}' \twoheadrightarrow p^*\mathcal{F}$ over U and X'. □

Proposition (4.11). *Let X be an algebraic stack and let $p\colon X' \to X$ be étale, representable, surjective and of finite presentation. Then X is pseudo-noetherian if and only if X' is pseudo-noetherian.*

Proof. The condition is necessary by definition. To show that it is sufficient, let $\mathbf{D} \subseteq \mathbf{E} = \mathbf{Stack}_{\mathrm{fp,\acute{e}t}/X}$ be the full subcategory with objects étale and finitely presented morphisms $Y \to X$ such that Y is pseudo-noetherian. By the definition of a pseudo-noetherian stack, the category $\mathbf{D}$ satisfies condition (D1) of Theorem (3.5). That $\mathbf{D}$ satisfies conditions (D2) and (D3) follows from Lemmas (4.9) and (4.10). Since $X' \in \mathbf{D}$ we conclude from Theorem (3.5) that $X \in \mathbf{D}$, i.e., that X is pseudo-noetherian. □

Corollary (4.12). *Every quasi-compact and quasi-separated Deligne–Mumford stack is pseudo-noetherian.*

Finally, we prove Theorem A, that is, that every stack of approximation type is pseudo-noetherian. We give the following slightly stronger version.

Theorem (4.13). *Let X be a pseudo-noetherian stack and let $X' \to X$ be of approximation type. Then X' is pseudo-noetherian. In particular, stacks of approximation type are pseudo-noetherian.*

Proof. As $X' \to X$ is of approximation type, there is by definition a surjective representable and finitely presented étale morphism $X'' \to X'$ such that $X'' \to X$ is of strict approximation type. By Proposition (4.8) we have that X'' is pseudo-noetherian and it follows that X' is pseudo-noetherian by Proposition (4.11). The last statement follows from the fact that $\operatorname{Spec} \mathbb{Z}$ is pseudo-noetherian. □

5. Finite coverings of stacks

In this section, we prove Theorem B, that is, that every quasi-compact stack X with quasi-finite and separated diagonal admits a finite surjective morphism of finite presentation from a *scheme* Z that is flat over a dense quasi-compact open substack $U \subseteq X$. Furthermore, if X is Deligne–Mumford, then there is such a Z which is étale over U.

Lemma (5.1) (Variant of Zariski's Main Theorem)**.** *Let $f\colon X \to Y$ be a representable quasi-finite and separated morphism of algebraic stacks such that $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is injective and integrally closed. Then f is an open immersion.*

Proof. As the question is fppf-local on Y and the integral closure commutes with smooth base change [LMB00, Prop. 16.2], we can assume that Y is an affine scheme. The result then follows from Zariski's Main Theorem [$\mathrm{EGA}_{\mathrm{IV}}$, 18.12.13 and 8.12.3]. □

Proof of Theorem B. Let $\pi\colon X' \to X$ be a separated and quasi-finite flat (resp. separated and quasi-compact étale) presentation by a scheme X', as exists by [Ryd11b, Thm. 7.1]. The separable fiber rank of π is constructible and lower semi-continuous [$\mathrm{EGA}_{\mathrm{IV}}$, Cor. 9.7.9, Prop. 15.5.9]. There is thus a quasi-compact open dense substack $U \subseteq X$ such that the separable rank is locally constant on U. Let $U = U_1 \amalg U_2 \amalg \cdots \amalg U_n$ be the decomposition into open and closed substacks such that π has constant separable fiber rank d over U_d. The theorem follows if we construct a scheme Z_d and a finite and finitely presented morphism $q\colon Z_d \to X$ such that $q|_{U_d}$ is flat (resp. étale) and surjective and such that $q^{-1}(U_k) = \emptyset$ for $k \neq d$.

To simplify notation, set $U = U_d$ and let $U' = \pi^{-1}(U)$. Let $(U'/U)^d = U' \times_U U' \times_U \cdots \times_U U'$ and let $V = \mathrm{SEC}_d(U'/U) \subseteq (U'/U)^d$ be the open subscheme given by the complement of the union of all diagonals. The structure morphism $V \to U$ is quasi-finite, flat, finitely presented and separated with fibers of separable rank $d!$. It follows that $V \to U$ is finite [$\mathrm{EGA}_{\mathrm{IV}}$, Prop. 15.5.9]. If π is étale, then $V \to U$ is also étale. Let $p\colon W \to X$ be the normalization of X in V. Then p is surjective and integral, the restriction $p|_U\colon W|_U \cong V \to U$ is flat and of finite presentation (resp. étale) and $p(W) = \overline{U}$. We will now show that W is a scheme.

Let $W' = W \times_X X'$ and $V' = V \times_U U'$; then W' is a scheme. We have d sections $s_i\colon V \to V'$ such that $\bigcup_i |s_i(V)| = |V'|$ as *sets*. Let $Y_i = \overline{s_i(V)}$ be the scheme-theoretic closure of the section s_i in W'. Then $|W'| = \bigcup_i |Y_i|$ since $\pi_W\colon W' \to W$ is flat. As W is integrally closed in V and $s_i(V) \cong V$ is schematically dense in Y_i, we have that $\mathcal{O}_W \to (\pi_W)_*\mathcal{O}_{Y_i}$ is injective and integrally closed. Thus, $(\pi_W)|_{Y_i}\colon Y_i \to W$ is an open immersion by Lemma (5.1). In particular, we have that $W = \bigcup_i Y_i$ is a *scheme*. The morphism $p\colon W \to X$ is integral and surjective and the restriction $p|_U\colon V \to U$ is finite, flat and finitely presented (resp. étale). The final step is to approximate p with a finitely presented morphism $p_\lambda\colon W_\lambda \to X$ such that W_λ is a scheme.

Let $\mathcal{A} = p_*\mathcal{O}_W$. We now use that X has the completeness property (Theorem A) and write $\mathcal{A}$ as a direct limit of finite and finitely presented algebras $\mathcal{A}_\lambda$ such that $(\mathcal{A}_\lambda)|_U = \mathcal{A}|_U$ and $\mathcal{A}_\lambda|_{U_k} = 0$ for all $k \neq d$, cf. Remark (4.4). Let $W_\lambda = \operatorname{Spec}_X(\mathcal{A}_\lambda)$; then $V = U \times_X W_\lambda$. Since $Y_i \to W$ is an open immersion, we have that $Y_i \hookrightarrow W'$ is a finitely presented closed immersion. By standard limit methods, there exists λ and a finitely presented closed immersion $(Y_i)_\lambda \hookrightarrow W'_\lambda = W_\lambda \times_X X'$ of schemes which pull-backs to $Y_i \hookrightarrow W'$. After increasing λ we can further assume, by Proposition (B.3), that the composition $(Y_i)_\lambda \hookrightarrow W'_\lambda \to W_\lambda$ is an open immersion and that $\coprod_i (Y_i)_\lambda \to W_\lambda$ is an open covering. In particular, we have that W_λ is a scheme. □

6. Properties stable under approximation

Let $X = \varprojlim_\lambda X_\lambda$ be an inverse limit of *finitely presented* stacks over S. In this section, we prove Theorem C for the system $(X_\lambda \to S)$, that is, we prove that if $X \to S$ has a certain property P, then so has $X_\lambda \to S$ for sufficiently large λ. This result is more elementary than the previous theorems and essentially independent of these. In fact, we only use the previous results when P is either "separated" or "proper with finite diagonal". Most of the properties are deduced by passing to the diagonal via the following lemma.

Lemma (6.1). *Let S be a quasi-compact algebraic stack. Let $X = \varprojlim_{\lambda \in L} X_\lambda$ be a limit of quasi-compact and quasi-separated morphisms (resp. quasi-separated morphisms of finite type) with affine bonding maps.*

(i) *The morphisms $g_\lambda\colon X \times_{X_\lambda} X \to X \times_S X$ are representable and of finite type (resp. of finite presentation). For every $\mu \geq \lambda$ the morphism $g_{\mu\lambda}\colon X \times_{X_\mu} X \to X \times_{X_\lambda} X$ is a closed immersion.*

(ii) *The inverse system $\{g_\lambda\}_{\lambda \in L}$ has limit $\Delta_{X/S}\colon X \to X \times_S X$.*

Proof. The morphism $X \times_{X_\lambda} X \to X \times_S X$ is a pull-back of the diagonal $\Delta_{X_\lambda/S}$ and hence representable and of finite type (resp. finite presentation). The bonding map $g_{\mu\lambda}$ is a pull-back of the diagonal of the affine morphism $X_\mu \to X_\lambda$.

Let L be the limit stack of the inverse system $\{g_\lambda\}$. By the universal property of the inverse limit L, the diagonals $X \hookrightarrow X \times_{X_\lambda} X$ factor through

L and the resulting map $X \to L$ is a monomorphism. Similarly, by the universal property of X, the two projections $\pi_1, \pi_2 \colon L \to X \times_{X_\lambda} X \to X$ coincide. It follows that $L \to X \times_{X_\lambda} X$ factors through Δ_{X/X_λ} and hence that $L = X$. $\square$

Proposition (6.2) ([TT90, Prop. C.6]). *Let S be a quasi-compact algebraic stack and let $X = \varprojlim_{\lambda \in L} X_\lambda$ be a limit of finitely presented S-stacks with affine bonding maps. If $X \to S$ is affine (resp. quasi-affine), then there is an index α such that $X_\lambda \to S$ is affine (resp. quasi-affine) for every $\lambda \geq \alpha$.*

Proof. We will first show the proposition under the assumption that the morphisms $X_\lambda \to S$ have affine double diagonals (e.g., separated diagonals).

The question is local on S in the fppf topology, so we can assume that S is an affine scheme. Let $\overline{X} = \operatorname{Spec}(\Gamma(\mathcal{O}_X))$ be the affine hull of X; recall that $X \to \overline{X}$ is a quasi-compact open immersion ($X = \overline{X}$ if X is affine). Since S is affine, we can write $\overline{X}$ as an inverse limit $\varprojlim_{\mu \in M} \overline{X}'_\mu$ of finitely presented affine S-schemes. By Remark (B.4), there is an index μ_0 and an open quasi-compact subscheme $X'_{\mu_0} \subseteq \overline{X}'_{\mu_0}$ with inverse image X in $\overline{X}$. We let $X'_\mu = X'_{\mu_0} \times_{\overline{X}'_{\mu_0}} \overline{X}'_\mu$ for all $\mu \geq \mu_0$. Then $X = \varprojlim_{\mu \in M} X'_\mu$ becomes a limit of finitely presented affine (resp. quasi-affine) S-schemes.

Let $\alpha_0 \in L$ be an index. By the functorial characterization of finitely presented morphisms, Proposition (B.1), there are indices $\alpha \in L$ and $\beta \in M$ and morphisms

$$X_\alpha \to X'_\beta \to X_{\alpha_0}$$

and after increasing α, we can assume that the composition coincides with the bonding map of the system (X_λ) and hence is affine. As X'_β has affine diagonal and X_{α_0} has affine double diagonal, it follows that $X'_\beta \to X_{\alpha_0}$ has affine diagonal and that $X_\alpha \to X'_\beta$ is affine. Thus, X_λ is affine (resp. quasi-affine) for every $\lambda \geq \alpha$.

For general X_λ, we at least know that the triple diagonal is affine (it is an isomorphism). Repeating the argument above we conclude that X_λ has affine diagonal for every $\lambda \geq \alpha$ and the proposition follows from the special case. $\square$

Corollary (6.3). *Let S be a quasi-compact* scheme *and let $X = \varprojlim_{\lambda \in L} X_\lambda$ be a limit of finitely presented S-stacks with affine bonding maps. If X is a scheme, then there is an index α such that X_λ is a scheme for every $\lambda \geq \alpha$.*

Proof. The question is Zariski-local on S so we can assume that S is affine. Choose an open affine covering $X = \bigcup_{i=1}^n U_i$. By Remark (B.4), there is an index λ and open subsets $U_{i,\lambda} \subseteq X_\lambda$ such that $U_i = U_{i,\lambda} \times_{X_\lambda} X$ for all i. After increasing λ we have that $U_{i,\lambda}$ is affine by Proposition (6.2). Finally, after further increasing λ we can assume that $X_\lambda = \bigcup_{i=1}^n U_{i,\lambda}$ and then X_λ is a scheme. $\square$

Proposition (6.4). *Let S be a quasi-compact algebraic stack and let $X = \varprojlim_\lambda X_\lambda$ be an inverse limit of finitely presented S-stacks such that the bonding maps $X_\mu \to X_\lambda$ are closed immersions for every $\mu \geq \lambda$. If $X \to S$ has one of the following properties:*

(i) *a monomorphism,*
(ii) *universally injective (i.e., "radiciel"),*
(iii) *representable,*
(iv) *unramified,*
(v) *quasi-finite,*
(vi) *finite,*
(vii) *a closed immersion,*
(viii) *an immersion;*

then there exists α such that $X_\lambda \to S$ has the corresponding property for all $\lambda \geq \alpha$.

If, in addition, the X_λ's and X are S-group spaces such that $X_\mu \hookrightarrow X_\lambda$ is a subgroup for every $\mu \geq \lambda$, then the same conclusion holds for the properties:

(ix) *abelian fibers,*
(x) *quasi-finite with linearly reductive fibers.*

Proof. As the properties are local in the fppf topology and S is quasi-compact, we can assume that S is an affine scheme. Note that $X \hookrightarrow X_\lambda$ is a closed immersion for every λ, so $X \to S$ is of finite type. It follows from Lemma (A.1) that properties (i)–(v) can be checked on fibers. Let P be one of these five properties or one of the properties (ix)–(x) for group spaces. We let $U_\lambda \subseteq |S|$ be the set of points $s \in |S|$ such that $(X_\lambda)_s \to \operatorname{Spec} \kappa(s)$ has property P. Then $U_\lambda \subseteq |S|$ is constructible by Propositions (A.3) and (A.4).

As a closed immersion has property P, it follows that $U_\lambda \subseteq U_\mu$ if $\lambda \leq \mu$. If $s \in |S|$ is any point, then as $X_s \to \operatorname{Spec} \kappa(s)$ is of finite type, we have that $X_s = (X_\lambda)_s$ for sufficiently large λ. It thus follows that $|S| = \bigcup U_\lambda$. As the constructible topology is quasi-compact, it follows that $U_\lambda = |S|$ for sufficiently large λ. This completes the demonstration of properties (i)–(v) and (ix)–(x).

Now assume that $X \to S$ is a closed immersion (resp. finite). By Proposition (6.2) we can assume that the maps $X_\lambda \to S$ are affine. Let $S = \operatorname{Spec} A$, $X_\lambda = \operatorname{Spec} B_\lambda$ and $X = \operatorname{Spec} B$. Choose an index λ and generators $b_1, b_2, \dots, b_n \in B_\lambda$. The image of b_i in B lifts to A (resp. satisfies a monic equation with coefficients in A). If $a_i \in A$ is a lifting, then the images of a_i and b_i coincides in B_μ (resp. the image of b_i in B_μ satisfies the monic equation) for some $\mu \geq \lambda$. As $B_\lambda \to B_\mu$ is surjective it follows that $A \to B_\mu$ is surjective (resp. finite). This settles properties (vii) and (vi).

If $X \to S$ is an immersion, then let $U \subseteq S$ be an open subscheme containing the image of X such that $X \to U$ is a closed immersion. As U is ind-constructible, it follows that $X_\lambda \to S$ factors through U for sufficiently large λ [EGA$_{\mathrm{IV}}$, Cor. 8.3.4]. Property (viii) thus follows from property (vii). □

Corollary (6.5). *Let S be a quasi-compact algebraic stack and let $X = \varprojlim_\lambda X_\lambda$ be an inverse limit of algebraic S-stacks of finite type with affine bonding maps. If the diagonal of $X \to S$ has one of the properties:*

(i) *a monomorphism,*
(ii) *unramified,*
(iii) *quasi-finite,*
(iv) *finite,*

(v) *a closed immersion,*
(vi) *an immersion,*
(vii) *affine,*
(viii) *quasi-affine,*
(ix) *separated,*
(x) *locally separated;*

then there exists α such that the diagonal of $X_\lambda \to S$ has the corresponding property for all $\lambda \geq \alpha$. In particular, if X/S has one of the properties: representable, representable and separated, representable and locally separated, relatively Deligne–Mumford, etc.; then so has X_λ/S.

If the inertia of $X \to S$ has one of the properties:

(xi) *finite,*
(xii) *abelian fibers,*
(xiii) *quasi-finite with linearly reductive fibers;*

then there exists α such that the inertia of $X_\lambda \to S$ has the corresponding property for all $\lambda \geq \alpha$.

Proof. Let P be one of the properties (i)–(viii). By Lemma (6.1), the diagonal $X \to X \times_S X$ is the inverse limit of the finitely presented morphisms $X \times_{X_\lambda} X \to X \times_S X$ and the bonding maps $X \times_{X_\mu} X \to X \times_{X_\lambda} X$ are closed immersions. It thus follows from Propositions (6.2) and (6.4) that if the diagonal of X/S has property P, then so has $X \times_{X_\lambda} X \to X \times_S X$ for sufficiently large λ. As $X \times_S X$ is the inverse limit of $X_\mu \times_S X_\mu$, it follows by standard limit results, Proposition (B.3), that $X_\mu \times_{X_\lambda} X_\mu \to X_\mu \times_S X_\mu$ has property P for sufficiently large $\mu \geq \lambda$. As the diagonal $X_\mu \to X_\mu \times_{X_\lambda} X_\mu$ is a closed immersion, it follows that the diagonal of X_μ/S has property P. For properties (ix) and (x) we reason as above, using that we have proven the Corollary for properties (v) and (vi).

Let P be one of the properties (xi)–(xiii). The pull-back of the inverse system $X \times_{X_\lambda} X \to X \times_S X$ along the diagonal $\Delta_{X/S}$ gives the inverse system $I_{X_\lambda/S} \times_{X_\lambda} X \to X$ with inverse limit the inertia $I_{X/S} \to X$. As the bonding maps in the first system are closed immersions, the bonding maps in the second system are closed subgroups. It thus follows from Proposition (6.4) that if the inertia $I_{X/S} \to X$ has property P then so has $I_{X_\lambda/S} \times_{X_\lambda} X \to X$ for all sufficiently large λ. By standard limit results, Proposition (B.3), it follows that $I_{X_\lambda/S} \times_{X_\lambda} X_\mu \to X_\mu$ has property P for all sufficiently large $\mu \geq \lambda$ and, a fortiori, so has $I_{X_\mu/S} \hookrightarrow I_{X_\lambda/S} \times_{X_\lambda} X_\mu \to X_\mu$. $\square$

Corollary (6.6). *Let S be a quasi-compact algebraic stack and let $X = \varprojlim_\lambda X_\lambda$ be an inverse limit of finitely presented S-stacks such that $X_\mu \to X_\lambda$ is a* closed immersion *for every $\mu \geq \lambda$. If $X \to S$ is proper with finite diagonal, then so is $X_\lambda \to S$ for all sufficiently large λ.*

Proof. The question is fppf-local on S so we can assume that S is affine. By Corollary (6.5) we can assume that $X_\lambda \to S$ has finite diagonal. Then there exists a scheme Z_λ and a finite and finitely presented surjective morphism $Z_\lambda \to X_\lambda$ by Theorem B. We let $Z_\mu = Z_\lambda \times_{X_\lambda} X_\mu$ for all $\mu > \lambda$ and $Z = Z_\lambda \times_{X_\lambda} X$. It is then enough to show that $Z_\mu \to S$ is proper for sufficiently large $\mu \geq \lambda$.

Since $Z_\lambda \to S$ is separated and of finite presentation (and S is affine), there is by Nagata's compactification theorem [Lüt93] a proper morphism $\overline{Z_\lambda} \to S$ and an open immersion $Z_\lambda \subseteq \overline{Z_\lambda}$. But $Z \to \overline{Z_\lambda}$ is then a closed immersion so it follows from Proposition (6.4) that $Z_\mu \to \overline{Z_\lambda}$ is a closed immersion for sufficiently large μ. This shows that $Z_\mu \to S$ is proper. □

Corollary (6.7). *Let S be a quasi-compact algebraic stack and let $X = \varprojlim_\lambda X_\lambda$ be an inverse limit of algebraic S-stacks of finite type with affine bonding maps. If $X \to S$ is separated, then there exists α such that $X_\lambda \to S$ is separated for every $\lambda \geq \alpha$.*

Proof. Reason as in the proof of Corollary (6.5) using Corollary (6.6). □

We have now proved Theorem C under the additional assumption that every morphism $X_\lambda \to S$ is of finite presentation and that S is a scheme in (iii). Indeed, this is Propositions (6.2), (6.4) and Corollaries (6.3), (6.5), (6.6) and (6.7). In the next section, we will deduce Theorem C from Theorem D and the following result.

Lemma (6.8). *Let $X \to S$ be a morphism of stacks. Let*

$$\{u_\lambda \colon X_\lambda \to S\}_{\lambda \in L} \quad \textit{and} \quad \{v_\mu \colon X'_\mu \to S\}_{\mu \in M}$$

be two inverse systems with limit $X \to S$ and bonding maps that are affine (resp. closed immersions). Assume that u_λ is quasi-compact and quasi-separated for all λ and that v_μ is of finite presentation for all μ. Further, assume that there exists $\alpha \in L$ and a factorization of u_α into an affine morphism (resp. closed immersion) $X_\alpha \to X_0$ followed by a morphism $u_0 \colon X_0 \to S$ of finite presentation.

Let P be a property of morphisms of stacks that is stable under composition with affine morphisms (resp. closed immersions). If v_μ has property P for sufficiently large μ, then so has u_λ for sufficiently large λ.

Proof. Since u_0 is of finite presentation, there is for sufficiently large μ a factorization $X \to X'_\mu \to X_0 \to S$ by Proposition (B.1). As $X \to X_\lambda \to X_0$ is affine and $X'_\mu \to X_0$ is of finite presentation, we can, by Proposition (6.2), assume that $X'_\mu \to X_0$ is affine after further increasing μ.

Note that $X \to X_0$ is the limit of the system $\{X_\lambda \to X_0\}_\lambda$ and that $X'_\mu \to X_0$ is of finite presentation. Thus, we can apply Proposition (B.1) and obtain, for sufficiently large λ, a factorization

$$X \to X_\lambda \to X'_\mu \to X_0.$$

Since $X_\lambda \to X_0$ is affine (resp. a closed immersion), so is $X_\lambda \to X'_\mu$ and it follows that $X_\lambda \to S$ has property P. □

7. Approximation of schemes and stacks

Recall that any stack of approximation type is pseudo-noetherian (Theorem A) and that any stack that is affine over a noetherian stack is pseudo-noetherian (Proposition 4.8). Conversely, it is possible that every pseudo-noetherian stack is affine over a noetherian stack. In this section, we prove Theorem D and, as a consequence, that stacks of approximation type are indeed affine over noetherian stacks.

Definition (7.1). Let S be a pseudo-noetherian stack and let $X \to S$ be a morphism of stacks. An *approximation* of X over S is a finitely presented S-stack X_0 together with an affine S-morphism $X \to X_0$. We say that X/S can be approximated if there exists an approximation of X over S.

Remark **(7.2).** In [$\mathrm{EGA_{IV}}$, 8.13.4], Grothendieck uses the term *essentially affine* for morphisms of schemes $X \to S$ that can be approximated.

Let S be pseudo-noetherian. Then Proposition (4.8) states that $X \to S$ has an approximation if and only if $X \to S$ is of strict approximation type. Moreover, if $X \to S$ has an approximation $X \to X_0 \to S$ then X and X_0 are pseudo-noetherian.

The following two propositions are analogues of properties (C1)–(C2) and (P1)–(P2) under the assumption that X/S can be approximated.

Proposition (7.3) (Completeness)**.** *Let X/S be an algebraic stack that can be approximated. Then*

(i) $X = \varprojlim_\lambda X_\lambda$ *such that $X_\lambda \to S$ is of finite type and $X \to X_\lambda$ is schematically dominant.*

(ii) $X = \varprojlim_\lambda X_\lambda$ *such that $X_\lambda \to S$ is of finite presentation.*

Proposition (7.4) (Presentation)**.** *Let X/S be an algebraic stack of* finite type *that can be approximated. Then*

(i) *There exists a finitely presented S-stack X_0 together with a closed immersion $X \hookrightarrow X_0$ over S.*

(ii) $X = \varprojlim_\lambda X_\lambda$ *such that $X_\lambda \to S$ is of finite presentation and the bonding map $X_\mu \to X_\lambda$ is a closed immersion for every $\mu \geq \lambda$.*

Proofs. Let $X \to X_0 \to S$ be an approximation and apply the completeness properties (C1)–(C2) and (P1)–(P2) on the affine morphism $f\colon X \to X_0$ (i.e., on the sheaf of $\mathcal{O}_{X_0}$-algebras $f_*\mathcal{O}_X$). $\square$

Remark **(7.5).** If $X \to S$ has an approximation and $U \subseteq X$ is a quasi-compact open substack, then by standard limit methods, cf. Remark (B.4), there exists an approximation $X_0 \to S$ of X and a quasi-compact open substack $U_0 \subseteq X_0$ such that $U = U_0 \times_{X_0} X$. We say that $(U_0 \subseteq X_0) \to S$ is an approximation of $(U \subseteq X) \to S$.

If $X \to S$ is affine, then X has a trivial approximation, namely S itself. At first, this hardly appears to be an "approximation" but the crucial point is that we assume that S is pseudo-noetherian. Then the statement that $X \to S$ can be approximated implies that X is the inverse limit of finitely presented and affine S-schemes.

Proposition (7.6). *Let S be a pseudo-noetherian stack. If X is a stack that is affine (resp. quasi-affine) over S, then X can be approximated by a stack that is affine (resp. quasi-affine) over S.*

Proof. The affine part is trivial, cf. the preceding discussion. If $f\colon X \to S$ is quasi-affine, let $\overline{X} = \mathrm{Spec}_S(f_*(\mathcal{O}_X))$ so that $X \to \overline{X}$ is a quasi-compact open immersion and $\overline{X} \to S$ is affine. By Remark (7.5), there is an approximation $(X \subseteq \overline{X}) \to (X_0 \subseteq \overline{X}_0) \to S$. The morphism $X_0 \to S$ is a quasi-affine approximation of $X \to S$. $\square$

The following proposition is an analogue of (E2).

Proposition (7.7) (Extension)**.** *Let $X \to S$ be a morphism of pseudo-noetherian stacks. Let $U \subseteq X$ be a quasi-compact open substack and let $U = \varprojlim_\lambda U_\lambda$ be an inverse limit of finitely presented S-stacks. If $X \to S$ can be approximated, then there exists an index α such that for any $\lambda \geq \alpha$, the approximation $U \to U_\lambda \to S$ extends to an approximation $(U \subseteq X) \to (U_\lambda \subseteq X_\lambda) \to S$.*

Proof. By Remark (7.5), there is an approximation $(U_0 \subseteq X_0) \to S$ of $(U \subseteq X) \to S$. As $U_0 \to S$ is finitely presented, the morphism $U \to U_0$ lifts to $U_\lambda \to U_0$ for sufficiently large λ by Proposition (B.1). This gives us the cartesian diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] \arrow[drr, phantom, "\square"] & U_\lambda \arrow[r] & U_0 \arrow[d] \\ X \arrow[rr] & & X_0. \end{tikzcd}$$

As $U \to U_0$ is affine, we have that $U_\lambda \to U_0$ is affine for sufficiently large λ by Proposition (6.2). By assumption X_0 has the completeness property and we can thus, using (E2), extend the diagram above to a cartesian diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U_\lambda \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U_0 \arrow[d] \\ X \arrow[r] & X_\lambda \arrow[r] & X_0 \end{tikzcd}$$

where $X_\lambda \to X_0$ is affine and finitely presented. The pair $(U_\lambda \subseteq X_\lambda) \to S$ is an approximation of $(U \subseteq X) \to S$. $\square$

We will now proceed with the étale dévissage that is used to show that morphisms of approximation type can be approximated.

Lemma (7.8). *Let $X \to S$ be a morphism of pseudo-noetherian stacks. Let $U \subseteq X$ be a quasi-compact open substack and let $p\colon X' \to X$ be a finitely presented étale neighborhood of $Z = X \setminus U$. If $U \to S$ and $X' \to S$ can be approximated, then so can $X \to S$.*

Proof. Let $U' = p^{-1}(U)$. Write U as an inverse limit $\varprojlim_\lambda U_\lambda$. For sufficiently large λ, there exists an étale morphism $U'_\lambda \to U_\lambda$ of finite presentation such that $U'_\lambda \times_{U_\lambda} U = U'$. By Proposition (7.7) we can, for sufficiently large λ, extend the approximation $U' \to U'_\lambda \to S$ to an approximation $(U' \subseteq X') \to (U'_\lambda \subseteq X'_\lambda) \to S$.

By Theorem (3.3) we have that X is the pushout of $U' \subseteq X'$ and $p|_U\colon U' \to U$. Let X_λ be the pushout of $U'_\lambda \subseteq X'_\lambda$ and $U'_\lambda \to U_\lambda$. This pushout exists by Theorem (3.4) and the morphism $X'_\lambda \amalg U_\lambda \to X_\lambda$ is étale and surjective. We have furthermore a 2-cartesian diagram

$$\begin{tikzcd} X' \amalg U \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X'_\lambda \amalg U_\lambda \arrow[d] \\ X \arrow[r] & X_\lambda \end{tikzcd}$$

by [Ryd11b, Prop. 3.2]. It follows that $X_\lambda \to S$ is of finite presentation and that $X \to X_\lambda$ is affine, so $X \to X_\lambda \to S$ is an approximation. □

Lemma (7.9). *Let $X \to S$ be a morphism of pseudo-noetherian stacks. Let $p\colon X' \to X$ be a representable étale and surjective morphism of finite presentation. If $X' \to S$ can be approximated, then so can $X \to S$.*

Proof. Let $\mathbf{D} \subseteq \mathbf{E} = \mathbf{Stack}_{\mathrm{fp,\acute{e}t}/X}$ be the full subcategory with objects étale and finitely presented morphisms $Y \to X$ such that $Y \to X \to S$ is of strict approximation type. As S is pseudo-noetherian, we have that $(Y \to X) \in \mathbf{D}$ if and only if $Y \to X \to S$ can be approximated (Proposition 4.8).

The category $\mathbf{D}$ satisfies condition (D1) of Theorem (3.5) by definition. That $\mathbf{D}$ satisfies conditions (D2) and (D3) follows from Proposition (2.12) and Lemma (7.8). Since $X' \in \mathbf{D}$, we thus conclude from Theorem (3.5) that $X \in \mathbf{D}$, i.e., that $X \to S$ can be approximated. □

Theorem (7.10). *Let S be a pseudo-noetherian stack and let $f\colon X \to S$ be a morphism of algebraic stacks. The following are equivalent:*

(i) *$X \to S$ is of approximation type;*
(ii) *$X \to S$ is of strict approximation type; and*
(iii) *$X \to S$ has an approximation.*

Proof. By definition we have that (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i). That (ii) $\Longrightarrow$ (iii) is Proposition (4.8) and that (i) $\Longrightarrow$ (ii) is Lemma (7.9). □

Proof of Theorem D. By Theorem (7.10) there exists an approximation $X \to X_0 \to S$. By Propositions (7.3) and (7.4), we can thus write X as an inverse limit of finitely presented morphisms $X_\lambda \to S$. The bonding maps $X_\mu \to X_\lambda$ are affine (resp. closed immersions) for general $X \to S$ (resp. for $X \to S$ of finite type). If $X \to S$ is integral, then, since S is pseudo-noetherian, we can arrange so that $X_\mu \to X_\lambda$ and $X_\lambda \to S$ are finite by the completeness property (C2) for the category of integral algebras.

Let P be one of the properties in (PA) or (PC). If $X \to S$ has P then $X_\lambda \to S$ has property P for all sufficiently large λ by Theorem C (finitely presented case). The statement on properties of $X \to \operatorname{Spec} \mathbb{Z}$ is an immediate consequence of the general form of Theorem C, proven below. □

We will now reduce the general case of Theorem C to the finitely presented case, proved in the previous section. This is done by a somewhat subtle bootstrapping process via Lemma (6.8) and the following three lemmas. The first lemma is Theorem C under the additional assumption that $X_\lambda \to S$ is of approximation type *fppf-locally over S*. The second lemma is the analogue of Proposition (B.3) but for morphisms of finite type. It is revisited in Theorem (8.10). The third lemma is Theorem C for diagonal and inertia properties.

Lemma (7.11). *Let S be a quasi-compact algebraic stack and let $\{X_\lambda \to S\}$ be an inverse system of quasi-compact and quasi-separated morphisms (resp. quasi-separated morphisms of finite type) of algebraic stacks with limit $X \to S$ and bonding maps that are affine (resp. closed immersions).*

(i) *Assume that $X_\lambda \to S$ is of approximation type fppf-locally over S and let P be one of the properties in* (PA) *(resp.* (PC)*). Then, if $X \to S$ has property P, so has $X_\lambda \to S$ for all sufficiently large λ.*
(ii) *Assume that S is Zariski-locally quasi-separated. Then, if X is a scheme, so is X_λ for all sufficiently large λ.*

Proof. The first claim is fppf-local on S so we can assume that $X \to S$ and $X_\lambda \to S$ are of approximation type. The second claim is Zariski-local on S so we can assume that S is quasi-separated and then replace S with $\operatorname{Spec} \mathbb{Z}$. As morphisms of schemes clearly are of approximation type, the second claim reduces to the first claim with P as the property "strongly representable".

By Theorem D, there are approximations $X = \varprojlim_\mu X'_\mu$ and $X_\alpha \to X_0 \to X$ where $X'_\mu \to S$ and $X_0 \to X$ are of finite presentation and the bonding maps, as well as $X_\alpha \to X_0$, are affine (resp. closed immersions). By the finitely presented case of Theorem C, it follows that $X'_\mu \to S$ has property P for sufficiently large μ. We conclude that $X_\lambda \to S$ has property P for sufficiently large λ by Lemma (6.8). □

Lemma (7.12). *Let S_0 be a quasi-compact algebraic stack, let $f_0 \colon X_0 \to S_0$ be a quasi-separated morphism of finite type and let $S = \varprojlim_\lambda S_\lambda$ be an inverse system of stacks that are affine over S_0. For every λ, let $f_\lambda \colon X_\lambda \to S_\lambda$ (resp. $f \colon X \to S$) be the base change of f_0 along $S_\lambda \to S_0$ (resp. $S \to S_0$). Let P be one of the properties of* (PA) *or* (PC)*. Assume that f_0 is of approximation type, fppf-locally over S_0. Then f has P if and only if f_λ has P for all sufficiently large λ.*

Proof. The question is fppf-local on S_0 so we can assume that S_0 is affine and that f_0 is of approximation type. By Theorem D, we can write $X_0 = \varprojlim_\mu X_0^\mu$ as an inverse limit of finitely presented morphisms $X_0^\mu \to S_0$ with bonding maps that are closed immersions. Then $X = \varprojlim_\mu X^\mu$ where $X^\mu = X_0^\mu \times_{S_0} S$. By the finitely presented case of Theorem C, it follows that $X^\mu \to S$ has property P for sufficiently large μ. We then apply Proposition (B.3) to $X_0^\mu \to S_0$ and deduce that $X_\lambda^\mu \to S_\lambda$ has property P for sufficiently large λ where $X_\lambda^\mu = X_0^\mu \times_{S_0} S_\lambda$. As property P is stable under composition with closed immersions, it follows that $f_\lambda \colon X_\lambda \to S_\lambda$ has property P. □

Lemma (7.13). *Let S be a quasi-compact algebraic stack and let $\{X_\lambda \to S\}_{\lambda \in L}$ be an inverse system of quasi-compact and quasi-separated morphisms with affine bonding maps $X_\mu \to X_\lambda$ and limit $X \to S$. Let P be one of the properties of* (PA) *or* (PC)*. If $\Delta_{X/S}$ has property P, then $\Delta_{X_\lambda/S}$ has property P for sufficiently large λ. If the inertia $I_{X/S}$ is finite (resp. abelian, resp. tame), then so is the inertia $I_{X_\lambda/S}$ for sufficiently large λ.*

Proof. The proof is almost identical to the proof of Corollary (6.5) replacing Propositions (6.4) and (B.3) with the two lemmas above. Recall that, by Lemma (6.1), the diagonal $\Delta_{X/S} \colon X \to X \times_S X$ is the inverse limit of the morphisms $X \times_{X_\lambda} X \to X \times_S X$ and that the bonding maps $X \times_{X_\mu} X \to X \times_{X_\lambda} X$ are closed immersions. Note that the morphisms $X \times_{X_\lambda} X \to X \times_S X$ are quasi-separated and of finite type but not necessarily of finite presentation. However, $X \times_{X_\lambda} X \to X \times_S X$ is representable and thus of approximation type, fppf-locally on the target. By Lemma (7.11), we deduce that $X \times_{X_\lambda} X \to X \times_S X$ has property P for sufficiently large λ.

By Lemma (7.12) we then deduce that $X_\mu \times_{X_\lambda} X_\mu \to X_\mu \times_S X_\mu$ has property P for all sufficiently large μ. As the bonding maps are affine, the diagonal Δ_{X_μ/X_λ} is a closed immersion. As property P is stable under composition with closed immersions, it follows that Δ_{X_μ} has property P for all sufficiently large μ.

The proof of the last statement about inertia is similar, cf. the proof of Corollary (6.5). □

Proof of Theorem C. We note that part (iii) is part (ii) of Lemma (7.11). The other parts are fppf-local on S so we can assume that S is affine.

Each property of (PA), except for affine and quasi-affine, corresponds to a property in (PA) or (PC) for the diagonal or a property for the inertia. Theorem C for these properties is thus an immediate consequence of Lemma (7.13).

Let P be the property quasi-finite. We may then assume that X_α has quasi-finite diagonal; hence there exists a quasi-finite flat presentation $U_\alpha \to X_\alpha$ [Ryd11b, Thm. 7.1]. It is enough to prove that $U_\lambda = U_\alpha \times_{X_\alpha} X_\lambda \to S$ is quasi-finite for sufficiently large λ. We can thus replace X_λ with U_λ and assume that X_λ is an algebraic space.

Each of the remaining properties of (PA) (affine, quasi-affine) and each property of (PC) except "quasi-finite" implies that the diagonal is quasi-finite and locally separated.

Thus, for all the remaining properties, we can assume that X_λ has quasi-finite and locally separated diagonal. Then $X_\lambda \to S$ is of approximation type (Corollary 2.7). We may now conclude the proof of Theorem C with Lemma (7.11). □

8. Applications

The first application is a generalization of Chevalley's affineness theorem to non-noetherian schemes and algebraic spaces. Also, we replace finite morphisms by integral morphisms. Partial generalizations of this type for schemes have been given by M. Raynaud [Ray68, Prop. 3.1] and B. Conrad [Con07, Cor. A.2].

Theorem (8.1) (Chevalley)**.** *Let X be an affine scheme, let Y be an algebraic space and let $f\colon X \to Y$ be an integral and surjective morphism. Then Y is affine.*

Proof. As X is quasi-compact and f is surjective it follows that Y is quasi-compact. As f is universally closed and surjective and X is separated, it also follows that Y is separated.

By Theorem D, the morphism $f\colon X \to Y$ has an approximation $X \to X_0 \to Y$ where $X_0 \to Y$ is finite and finitely presented and X_0 is affine. Replacing X with X_0, we can thus assume that f is finitely presented.

By Theorem D, we can write Y as an inverse limit of noetherian algebraic spaces $(Y_\lambda)_\lambda$ such that $Y \to Y_\lambda$ is affine for every λ. Since f is finitely presented, there is, by standard limit methods, for sufficiently large λ, a finite surjective morphism $f_\lambda\colon X_\lambda \to Y_\lambda$ that pull-backs to $f\colon X \to Y$, cf. Appendix B. After increasing λ further, we can also assume that X_λ is affine by Theorem C. By Chevalley's theorem for finite morphisms between noetherian algebraic spaces [Knu71, Thm. III.4.1], it now follows that Y_λ is affine and hence that Y is affine. □

Corollary (8.2). *Let X be an algebraic space. If X_{red} is a scheme (resp. a quasi-affine scheme, resp. an affine scheme), then so is X.*

Proof. If X_{red} is an affine scheme, then it follows by Chevalley's theorem that X is an affine scheme since $X_{\mathrm{red}} \hookrightarrow X$ is finite and surjective. If X_{red} is a scheme, then there is an open covering $X = \bigcup U_i$ such that the $(U_i)_{\mathrm{red}}$ are affine and we conclude that the U_i are affine and that X is a scheme. If X_{red} is quasi-affine, then there exists an approximation $X_{\mathrm{red}} \hookrightarrow X_0 \hookrightarrow X$ where X_0 is quasi-affine and $X_0 \hookrightarrow X$ is a finitely presented closed immersion (Theorem D). We have seen that X is a scheme so [$\mathrm{EGA}_{\mathrm{II}}$, Prop. 4.5.13] applies and shows that X is quasi-affine. □

Theorem (8.3). *Let X and S be quasi-compact stacks with quasi-finite and separated diagonals. Let $f\colon X \to S$ be a universally closed, separated and quasi-compact morphism. Then f factors through an integral surjective morphism $X \to X'$ followed by a proper morphism $X' \to S$ with finite diagonal.*

Proof. By Theorem D there is an approximation $X \to X_0 \to S$ where $X \to X_0$ is affine and $X_0 \to S$ is of finite presentation with finite diagonal. It follows that $X \to X_0$ is universally closed and thus integral [$\mathrm{EGA}_{\mathrm{IV}}$, Prop. 18.12.8]. Let $X' \hookrightarrow X_0$ be the schematic image of $X \to X_0$. Then $X \to X'$ is surjective and $X' \to S$ is universally closed, hence proper. □

As an amusing corollary, we see that the finiteness assumption in Chevalley's theorem on the fiber dimension can be removed.

Corollary (8.4) (Chevalley)**.** *Let S be an algebraic stack and let $f\colon X \to S$ be a universally closed, separated and quasi-compact morphism with finite diagonal. Then, the fibers of f are finite-dimensional and the function $s \mapsto \dim(f^{-1}(s))$ is upper semi-continuous.*

Proof. Using Theorem (8.3) and Theorem B we can assume that f is proper and strongly representable. This case follows from [$\mathrm{EGA}_{\mathrm{IV}}$, Cor. 13.1.5]. □

The following theorem settles a question of Grothendieck [$\mathrm{EGA}_{\mathrm{IV}}$, Rem. 18.12.9]. This was also the original motivation for this paper.

Theorem (8.5). *Let $f\colon X \to S$ be a morphism of algebraic stacks. Then f is integral if and only if f is universally closed, separated and has affine fibers.*

Proof. Taking a presentation, we can assume that S is affine. The necessity follows from [$\mathrm{EGA}_{\mathrm{II}}$, Cor. 6.1.10] so assume that f is universally closed, separated and has affine fibers. Note that f is representable and quasi-compact, since the fibers of f are quasi-compact and f is closed. Thus, by Theorem (8.3), there is a factorization of f into an integral surjective morphism $X \to X'$ followed by a proper and representable morphism $X' \to$

S. Chevalley's theorem (8.1) then shows that the fibers of $X' \to S$ are affine and hence finite. As a quasi-finite and proper morphism is finite [LMB00, Cor. A.2.1], the theorem follows. $\square$

The following variant of Zariski's Main Theorem generalizes [EGA$_{\text{IV}}$, Cor. 18.12.13], [Knu71, Thm. II.6.15] and [LMB00, Thm. 16.5].

Theorem (8.6) (Zariski's Main Theorem)**.** *Let S be an algebraic stack and let $f\colon X \to S$ be a representable, quasi-finite and separated morphism. Then*

(i) *There is a factorization $X \to X' \to S$ of f where $X \to X'$ is an open immersion and $X' \to S$ is integral.*

(ii) *If S is pseudo-noetherian (or at least has the completeness property), then there exists a factorization as above with $X' \to S$ finite. If, in addition, f is of finite presentation, we can choose $X' \to S$ to be of finite presentation.*

Note that (ii) is satisfied if S is noetherian or of approximation type, e.g., quasi-compact with quasi-finite and locally separated diagonal.

Proof of Theorem (8.6)*.* (i) (cf. [LMB00, Thm. 16.5]) The integral closure of $\mathcal{O}_S \to f_*\mathcal{O}_X$ is a quasi-coherent $\mathcal{O}_S$-subalgebra $\mathcal{A} \subseteq f_*\mathcal{O}_X$. Let X' be the spectrum of $\mathcal{A}$. This gives a factorization $X \to X' \to S$ where the first morphism is quasi-finite, representable, separated, schematically dominant and integrally closed and the second morphism is integral. It follows that $X \to X'$ is an open immersion by Lemma (5.1).

(ii) Write $X' = \varprojlim_\lambda X'_\lambda$ as a limit of finite and finitely presented morphisms $X'_\lambda \to S$. For sufficiently large λ, there exists, by Remark (B.4), an open substack $X_\lambda \subseteq X'_\lambda$ such that $X = X_\lambda \times_{X'_\lambda} X'$. As f is of finite type so is $X \to X_\lambda$ and thus we have that $X \to X_\lambda$ is finite. If f is of finite presentation, then so is $X \to X_\lambda$. Since X'_λ has the completeness property, it has the extension property (E2) for the category of integral algebras. This gives the existence of a cartesian diagram

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X_\lambda \arrow[d] \\ X' \arrow[r] & X'' \arrow[r] & X'_\lambda \end{tikzcd}$$

where $X'' \to X'_\lambda$ is finite (resp. finite and of finite presentation). The requested factorization is $X \to X'' \to S$. $\square$

Theorem (8.7) (Serre's criterion)**.** *Let S be a quasi-compact algebraic stack with quasi-finite and separated diagonal and let $f\colon X \to S$ be a representable quasi-compact and quasi-separated morphism. Then f is affine if and only if $f_*\colon \mathbf{QCoh}(X) \to \mathbf{QCoh}(S)$ is exact (and if and only if f_* is faithfully exact).*

Proof. If f is affine, then f_* is faithfully exact. Indeed, this can be checked on a presentation $g\colon S' \to S$ since g^* is faithfully exact. Conversely, assume that f_* is exact. Let $g\colon S' \to S$ be a finite surjective morphism with S' a scheme as in Theorem B and let $f'\colon X' \to S'$ be the pull-back of f. Then since g_* is faithfully exact it follows that f'_* is exact.

Let $h\colon X'' \to X'$ be a finite surjective morphism with X' a scheme, cf. Theorem B. Then f'_*h_* is exact and it follows from Serre's criterion for schemes that $f' \circ h\colon X'' \to S'$ is affine [EGA$_{\text{II}}$, Cor. 5.2.2], [EGA$_{\text{IV}}$, 1.7.18]. In particular, we have that $g \circ f' \circ h$ is affine. It then follows from Chevalley's theorem (8.1) that f is affine. □

Using [Alp13, Prop. 3.10 (vii)], one may strengthen Theorem (8.7), only requiring that S is a, not necessarily quasi-compact, algebraic stack with quasi-affine diagonal.

The following forms of Chow's lemma generalize [LMB00, Cor. 16.6.1] and [RG71, Cor. 5.7.13]. Note that the hypothesis that $\mathscr{X}/S$ is separated is missing in the statement of [LMB00, Cor. 16.6.1].

Theorem (8.8) (Chow's lemma). *Let S be a quasi-compact and quasi-separated algebraic space, let X be a quasi-compact stack with quasi-finite and separated diagonal and let $f\colon X \to S$ be a morphism of finite presentation. Then there exists a commutative diagram*

$$\begin{tikzcd} X \arrow[d, "f"] & X' \arrow[l, "p"'] \arrow[d, "g"] \\ S & P \arrow[l, "\pi"'] \end{tikzcd}$$

of finitely presented morphisms such that:

(i) *X' is a scheme;*
(ii) *π is projective;*
(iii) *p is proper, strongly representable and surjective; and*
(iv) *g is strongly representable and étale (but not necessarily separated).*

If f is separated, then g can be chosen to be an open immersion (so $\pi \circ g$ is quasi-projective).

Proof. By Theorem D, we can assume that S is noetherian. Replacing X with a finite cover as in Theorem B, we can assume that X is a scheme. The result then follows from [RG71, Cor. 5.7.13]. □

We have the following variant of the previous result which is more in the spirit of the usual Chow's lemma for schemes. In this statement we can also drop the finite presentation hypothesis.

Theorem (8.9) (Chow's lemma). *Let S be a quasi-compact and quasi-separated algebraic space, let X be a quasi-compact stack with quasi-finite and separated diagonal and a* finite number of irreducible components. *Let $f\colon X \to S$ be a morphism of finite type. Then there exists a commutative diagram*

$$\begin{tikzcd} X \arrow[d, "f"] & X' \arrow[l, "p"'] \arrow[d, "g"] \\ S & P \arrow[l, "\pi"'] \end{tikzcd}$$

such that:

(i) *X' is a scheme;*
(ii) *π is projective;*

(iii) *p is proper, strongly representable and surjective and there exists a dense open substack $U \subseteq X$ such that $p|_U$ is finite, flat and finitely presented;*
(iv) *g is étale and strongly representable.*

Moreover, if f is separated, then g can be chosen as an open immersion and if X is Deligne–Mumford, then $p|_U$ can be taken to be étale.

Proof. Replacing X with a finite generically étale (resp. generically flat) cover as in Theorem B, we can assume that X is a scheme. The result then follows from [RG71, Cor. 5.7.13]. □

The following theorem partly generalizes Proposition (B.3) from finitely presented morphisms to quasi-separated morphisms of finite type.

Theorem (8.10). *Let S_0 be a quasi-compact algebraic stack and let $S = \varprojlim_\lambda S_\lambda$ be the limit of an inverse system of stacks that are affine over S_0. Let α be an index and let $f_\alpha\colon X_\alpha \to S_\alpha$ be a quasi-separated morphism of finite type. For every $\lambda > \alpha$, let $f_\lambda\colon X_\lambda \to S_\lambda$ (resp. $f\colon X \to S$) be the base change of f_α along $S_\lambda \to S_\alpha$ (resp. $S \to S_\alpha$). Let P be one of the properties of* (PA) *or* (PC)*. Then f (resp. Δ_f) has property P if and only if f_λ (resp. Δ_{f_λ}) has property P for all sufficiently large λ's.*

Proof. The question is fppf-local on S_0 so we can assume that S_0 is affine. Note that, by assumption, the diagonal Δ_{f_λ} is of finite presentation for every λ. If the property is a property of the diagonal (e.g., the property "separated" corresponds to the property "proper" for the diagonal) we deduce the theorem from Proposition (B.3) applied to Δ_{f_λ}.

If P is quasi-finite, then we may find a quasi-finite flat presentation $p\colon U \to X$ [Ryd11b, Thm. 7.1] with U affine. By standard limit methods there is, for sufficiently large λ, an affine scheme U_λ and a quasi-finite flat presentation $p_\lambda\colon U_\lambda \to X_\lambda$ that is pulled back to p. It is enough to show that $U_\lambda \to S_\lambda$ is quasi-finite for sufficiently large λ, so we can replace α with λ, X_α with U_λ and assume that X_α is affine.

If P is any of the other remaining properties, then the diagonal is quasi-finite and locally separated. We can thus apply Proposition (B.3) to deduce that, for sufficiently large λ, the diagonal Δ_{f_λ} is quasi-finite and locally separated. After increasing α we can thus assume that X_α is of global type and hence can be approximated.

The theorem now follows from Lemma (7.12). □

Theorem (8.10) does not hold for properties that are not stable under closed immersions such as: isomorphism, open immersion, étale, finite presentation, surjective, flat, smooth, universally subtrusive, universally open. An easy counter-example is furnished by taking $S_0 = \operatorname{Spec}(k[x_1, x_2, \dots])$ and $S = X_0$ as the inclusion of the origin.

Recall that a morphism $f\colon X \to Y$ is of *constructible finite type* if f is of finite type and quasi-separated, and for any morphism $Y' \to Y$, the base change $f'\colon X \times_Y Y' \to Y'$ maps ind-constructible subsets onto ind-constructible subsets [Ryd11a, App. D]. The following result was surmised in *loc. cit.*.

Proposition (8.11). *Let Y be a pseudo-noetherian stack and let $f\colon X \to Y$ be a morphism of approximation type (e.g., let X and Y be stacks of global type). Then f is of constructible finite type if and only if f can be factored as a nil-immersion $X \hookrightarrow X'$ followed by a finitely presented morphism $X' \to Y$.*

Proof. As nil-immersions and finitely presented morphisms are of constructible finite type the condition is sufficient. To see that it is necessary write $X = \varprojlim_\lambda X_\lambda$ as an inverse limit of finitely presented morphisms $X_\lambda \to Y$ with closed immersions as bonding maps. By assumption $X \hookrightarrow X_\lambda$ is of constructible finite type so $|X| \subseteq |X_\lambda|$ is constructible. It follows that $X \hookrightarrow X_\lambda$ is bijective for sufficiently large λ by [EGA$_{\mathrm{IV}}$, Cor. 8.3.5]. $\square$

Appendix A. Constructible properties

In this appendix, we extend standard results on constructible properties for schemes to algebraic stacks. We also show that if $G \to S$ is a group scheme, then the locus of points $s \in S$ with an abelian (resp. a finite and linearly reductive) fiber G_s, is open.

Lemma (A.1). *Let Y be an algebraic space. Let $f\colon X \to Y$ be a morphism between algebraic stacks that is locally of finite type. For a point $y \in |Y|$, let $f_y\colon X_y \to \operatorname{Spec} \kappa(y)$ denote the fiber. Then*

(i) *f is representable if and only if the fiber f_y is representable for every $y \in |Y|$;*
(ii) *f is a monomorphism if and only if the fiber f_y is a monomorphism for every $y \in |Y|$ (i.e., X_y is either empty or $\kappa(y)$-isomorphic to $\operatorname{Spec} \kappa(y)$ for every $y \in |Y|$); and*
(iii) *f is unramified if and only if the fiber f_y is unramified for every $y \in |Y|$.*

Proof. (iii) is [Ryd11a, Prop. B.3]. The necessity of (i) and (ii) is clear. We begin with showing the sufficiency of (ii) under the additional assumption that f is representable. If f_y is a monomorphism for every $y \in |Y|$, then f is unramified by (iii) and, in particular, Δ_f is an open immersion. As f is universally injective, we have that Δ_f is bijective and hence an isomorphism, i.e., f is a monomorphism.

Now assume that f_y is representable for every $y \in |Y|$. Then $(\Delta_{X/Y})_y$ is a monomorphism for every $y \in |Y|$ and hence $\Delta_{X/Y}$ is a monomorphism by the special case of (ii). This shows that f is representable. The general case of (ii) now follows from (i) and the special case of (ii). $\square$

The following lemma generalizes [EGA$_{\mathrm{IV}}$, Prop. 9.2.6.1] to non-representable morphisms.

Lemma (A.2). *Let S be an algebraic space and let X be an algebraic stack of finite presentation over S. The function $|S| \to \mathbb{Z}$ defined by $s \mapsto \dim X_s$ is constructible.*

Proof. Let $f\colon X \to S$ denote the structure morphism. Let $x\colon \operatorname{Spec} k \to X$ be a point and let $s = f \circ x$. The dimension $\dim_x X_s$ only depends on the image of the point x in the topological space $|X|$. This gives a function $\dim_{X/S}\colon |X| \to \mathbb{Z}$. Let $p\colon U \to X$ be a smooth presentation. Then

$\dim_{U/S}\colon |U| \to \mathbb{N}$ is constructible [$\mathrm{EGA}_{\mathrm{IV}}$, Prop. 9.9.1] (and upper semi-continuous) and $\dim_{U/X}\colon |U| \to \mathbb{N}$ is locally constant. Thus $\dim_{X/S} \circ |p| = \dim_{U/S} - \dim_{U/X}$ is a constructible (and upper semi-continuous) function. The set $f\bigl(\dim_{X/S}^{-1}(d)\bigr) = (f \circ p)\bigl((\dim_{X/S} \circ |p|)^{-1}(d)\bigr)$ is thus constructible by Chevalley's theorem [$\mathrm{EGA}_{\mathrm{I}}$, Thm. 7.1.4]. Since

$$\dim X_s = \sup_{x \in |X_s|} \dim_x X_s = \sup_d \Bigl\{ d \ : \ s \in f\bigl(\dim_{X/S}^{-1}(d)\bigr) \Bigr\}$$

it follows that $s \mapsto \dim X_s$ is constructible. □

Proposition (A.3). *Let S be an algebraic space and let $f\colon X \to Y$ be a morphism between algebraic stacks of finite presentation over S. Let P be one of the following properties of a morphism:*

(i) *monomorphism,*
(ii) *universally injective (i.e., "radiciel"),*
(iii) *surjective,*
(iv) *isomorphism,*
(v) *representable,*
(vi) *unramified,*
(vii) *flat,*
(viii) *étale,*
(ix) *quasi-finite,*
(x) *has quasi-finite diagonal.*

Then, the set of points $s \in |S|$ such that $f_s\colon X_s \to Y_s$ has P is constructible.

Proof. The question is fppf-local on S so we can assume that S is affine. We may also replace Y with a presentation and assume that Y is affine. When f is strongly representable, the proposition holds by [$\mathrm{EGA}_{\mathrm{IV}}$, Props. 9.6.1, 11.2.8 and 17.7.11].

If f is representable, then let $X' \to X$ be an étale presentation with X' a scheme. The corresponding result for $X' \to X \to Y$ implies the result for f and all properties with the exception of (i) monomorphism, (ii) universally injective and (iv) isomorphism. The locus where f is a monomorphism (resp. universally injective) coincides with the locus where Δ_f is an isomorphism (resp. surjective) and this locus is constructible (since Δ_f is strongly representable). This settles properties (i) and (ii). Finally, f is an isomorphism if and only if f is a surjective étale monomorphism, so property (iv) is constructible.

For general f, we can now deduce that the proposition holds for the properties: (i) monomorphism, (ii) universally injective, (v) representable, (vi) unramified and (x) quasi-finite diagonal; by considering the corresponding properties for the diagonal: (iv) isomorphism, (iii) surjective, (i) monomorphism, (viii) étale, and (ix) quasi-finite. Properties (iii) surjective and (vii) flat, follow by taking a presentation $X' \to X$. Property (viii) étale, is the conjunction of properties (vi) unramified and (vii) flat. As before, property (iv) isomorphism, is the conjunction of properties (i), (iii) and (viii).

Property (ix) quasi-finite can be checked on fibers and we can thus, by Chevalley's Theorem [$\mathrm{EGA}_{\mathrm{I}}$, Thm. 7.1.4], assume that $S = Y$. The set in question is then the set where the fibers of f and Δ_f both have dimension zero. This set is constructible by Lemma (A.2). □

Proposition (A.4). *Let S be an algebraic space and let G be an S-group space of finite presentation (i.e., a group object in the category of algebraic spaces). The set of points $s \in |S|$ such that G_s is abelian (resp. finite and linearly reductive) is constructible.*

Proof. The question is fppf-local on S so we can assume that S is a scheme. Let G act on itself by conjugation and let $\rho\colon G \times_S G \to G$ be the corresponding morphism, pointwise given by $(g,h) \mapsto ghg^{-1}$. As the diagonal $\Delta\colon G \to G \times_S G$ is of finite presentation, the subset $Z \subseteq |G \times_S G|$ where $\rho = \pi_2$ is constructible. As the structure morphism $p\colon G \times_S G \to S$ is of finite presentation, it follows that the subset $W = S \setminus p(G \times_S G \setminus Z)$, of points $s \in |S|$ such that G_s is abelian, is constructible.

For a group scheme $H \to \operatorname{Spec} k$, we let $E(H,k)$ be the property that $H \to \operatorname{Spec} k$ is finite and linearly reductive, or, equivalently, that $H \to \operatorname{Spec} k$ is *locally well-split* [AOV08, Prop. 2.10]. This property is stable under field extensions k'/k. Let E be the set of points $s \in |S|$ such that $E(G_s, \kappa(s))$ holds. We have to show that E is constructible. By [EGA$_{\mathrm{IV}}$, Prop. 9.2.3], it is enough to show that if S is an integral noetherian scheme, then there is an open dense subset $U \subseteq S$ that is contained in either E or $S \setminus E$.

To show this we can replace S with an open dense subset such that $G \to S$ becomes flat [EGA$_{\mathrm{IV}}$, Thm. 11.1.1]. Moreover, as the property of having finite fibers is constructible, we can assume that $G \to S$ is quasi-finite. After replacing S with an open dense subset, we can further assume that $G \to S$ is finite. Then E is open by [AOV08, Lem. 2.13] and thus either E is dense or empty. □

Appendix B. Standard limit results

In this appendix, we generalize the standard limit methods for schemes in [EGA$_{\mathrm{IV}}$, §8] to algebraic stacks. This has been done in [LMB00, Props. 4.15, 4.18] (also see [Ols06, Prop. 2.2]) for algebraic stacks over inverse systems of *affine* schemes. In this appendix, we allow inverse system of algebraic stacks. As elsewhere, we do not insist that the diagonal of an algebraic stack is separated. All inverse systems are assumed to be filtered and to have affine bonding maps so their inverse limits exist in the category of algebraic stacks.

We begin with the functorial characterization of morphisms that are locally of finite presentation, cf. [EGA$_{\mathrm{IV}}$, Prop. 8.14.2].

Proposition (B.1). *Let $f\colon Y \to S$ be a morphism of algebraic stacks. The following are equivalent.*

(i) *f is locally of finite presentation.*
(ii) *For every inverse system $\{g_\lambda\colon X_\lambda \to S\}$ of quasi-compact and quasi-separated stacks X_λ with limit $g\colon X \to S$ the functor*

$$\varinjlim_\lambda \mathbf{Hom}_S(X_\lambda, Y) \to \mathbf{Hom}_S(X, Y)$$

is an equivalence of categories.

(iii) *As* (ii) *but with* X_λ *affine for every* λ.

Proof. Clearly (ii) $\Longrightarrow$ (iii). That (i) $\Longleftrightarrow$ (iii) is [LMB00, Prop. 4.15 (i)]. Let us show that (iii) $\Longrightarrow$ (ii) which essentially is the proof of [LMB00, Prop. 4.18 (i)]. After making the base change $X_\lambda \to S$ for some λ, we can assume that the X_λ's are affine over S. Let $U_0 \to S$ be a presentation and let $U_\lambda = U_0 \times_S X_\lambda$; then $U_\lambda \to X_\lambda$ is a presentation. Let $(U_\lambda/X_\lambda)^i = U_\lambda \times_{X_\lambda} \cdots \times_{X_\lambda} U_\lambda$ denote the i^{th} fiber product. The category $\mathbf{Hom}_S(X_\lambda, Y)$ is equivalent to the category given by the cosimplicial diagram of categories

$$\mathbf{Hom}_S(U_\lambda, Y) \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}\; \mathbf{Hom}_S\big((U_\lambda/X_\lambda)^2, Y\big) \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}\; \mathbf{Hom}_S\big((U_\lambda/X_\lambda)^3, Y\big)$$

(cf. *loc. cit.*) and this construction commutes with filtered colimits. It is therefore enough to show the proposition after replacing X_λ with $(U_\lambda/X_\lambda)^i$ for $i = 1, 2, 3$.

Firstly, assume that S is a separated algebraic space, and choose a presentation $U_0 \to S$ with U_0 affine. Then the fiber products $(U_\lambda/X_\lambda)^i$ are affine for $i = 1, 2, 3$ and we are done in this case. Secondly, assume that S is an algebraic space. Then $(U_\lambda/X_\lambda)^i$ are separated algebraic spaces and this case follows from the previous. Thirdly, assume that S is a general algebraic stack. Then $(U_\lambda/X_\lambda)^i$ are algebraic spaces and this settles the final case. $\square$

Proposition (B.2). *Let* S_0 *be an algebraic stack and let* $S = \varprojlim_\lambda S_\lambda$ *be an inverse limit of stacks that are affine over* S_0.

(i) *Let* $X_0 \to S_0$ *and* $Y_0 \to S_0$ *be morphisms of stacks and let*

$$X_\lambda = X_0 \times_{S_0} S_\lambda, \quad Y_\lambda = Y_0 \times_{S_0} S_\lambda,$$
$$X = X_0 \times_{S_0} S, \quad Y = Y_0 \times_{S_0} S$$

for every λ. *Suppose that* X_0 *is quasi-compact and quasi-separated and that* $Y_0 \to S_0$ *is locally of finite presentation. Then, the functor*

$$\varinjlim_\lambda \mathbf{Hom}_{S_\lambda}(X_\lambda, Y_\lambda) \to \mathbf{Hom}_S(X, Y)$$

is an equivalence of categories.

(ii) *Suppose that* S_0 *is quasi-compact and quasi-separated. Let* $X \to S$ *be a morphism of finite presentation. Then, there exists an index* α, *an algebraic stack* X_α *of finite presentation over* S_α *and an* S-*isomorphism* $X_\alpha \times_{S_\alpha} S \to X$.

Proof. Note that $\mathbf{Hom}_{S_\lambda}(X_\lambda, Y_\lambda) = \mathbf{Hom}_{S_0}(X_\lambda, Y_0)$ and $\mathbf{Hom}_S(X, Y) = \mathbf{Hom}_{S_0}(X, Y_0)$. The first statement thus follows from Proposition (B.1) with $S = S_0$ and $Y = Y_0$.

(ii) When S_0 and X are schemes, this is [EGA_{IV}, Thm. 8.8.2 (ii)]. The extension to the case where X is an algebraic space is not difficult, cf. [LMB00, Prop. 4.18]. For the general case, choose a presentation $V_0 \to S_0$ with V_0 affine and let $V_\lambda = V_0 \times_{S_0} S_\lambda$ and $V = V_0 \times_{S_0} S$. Also choose a presentation $U \to X \times_S V$ and let $R = U \times_X U$. Then X is the quotient of the groupoid $[R \rightrightarrows U]$. Consider R as a $V \times_S V$-space.

Applying the case with algebraic spaces, there is an index λ and algebraic spaces U_λ and R_λ of finite presentation over V_λ and $V_\lambda \times_{S_\lambda} V_\lambda$ such that their pull-backs to V and $V \times_S V$ are isomorphic to $U \to V$ and $R \to V \times_S V$. By composition, we obtain finitely presented morphisms $U_\lambda \to S_\lambda$ and $R_\lambda \to S_\lambda$

such that their pull-backs are isomorphic to $U \to S$ and $R \to S$. Note that the last statement takes place in the 1-category $\mathbf{AlgSp}_{/S}$ and that isomorphic signifies that there are 2-commutative diagrams

$$\begin{tikzcd} U_\lambda \times_{S_\lambda} S \arrow[r, "\cong"] \arrow[rr, bend right, "\pi_2"', "\Downarrow" description] & U \arrow[r] & S \end{tikzcd} \qquad \begin{tikzcd} R_\lambda \times_{S_\lambda} S \arrow[r, "\cong"] \arrow[rr, bend right, "\pi_2"', "\Downarrow" description] & R \arrow[r] & S. \end{tikzcd}$$

By (i), we can, for sufficiently large λ, find morphisms such that we obtain a groupoid $R_\lambda \rightrightarrows U_\lambda$ in the category $\mathbf{AlgSp}_{/S_\lambda}$ which pull-backs to the groupoid $R \rightrightarrows U$ in the category $\mathbf{AlgSp}_{/S}$. For sufficiently large λ, we can also assume that the morphisms $s, t\colon R_\lambda \to U_\lambda$ are smooth. Indeed, this can be checked on an étale presentation of the algebraic space R_λ so we can apply [$\mathrm{EGA}_{\mathrm{IV}}$, Prop. 17.7.8].

Let $X_\lambda = [R_\lambda \rightrightarrows U_\lambda]$ be the quotient stack. Then, there is an induced finitely presented morphism $X_\lambda \to S_\lambda$, unique up to unique 2-isomorphism, such that the pull-back $X_\lambda \times_{S_\lambda} S \to S$ is isomorphic to $X \to S$. $\square$

Proposition (B.3). *Let S_0 be a quasi-compact algebraic stack and let $S = \varprojlim_\lambda S_\lambda$ be an inverse limit of stacks that are affine over S_0. Let α be an index and let $f_\alpha\colon X_\alpha \to Y_\alpha$ be a morphism between stacks of finite presentation over S_α. For every $\lambda > \alpha$, let $f_\lambda\colon X_\lambda \to Y_\lambda$ (resp. $f\colon X \to Y$) be the base change of f_α along $S_\lambda \to S_\alpha$ (resp. $S \to S_\alpha$). Let P be one of the following properties of a morphism:*

(i) *representable,*
(ii) *a monomorphism,*
(iii) *an isomorphism,*
(iv) *an immersion,*
(v) *a closed immersion,*
(vi) *an open immersion,*
(vii) *universally injective (i.e., "radiciel"),*
(viii) *a universal homeomorphism,*
(ix) *surjective,*
(x) *flat,*
(xi) *universally subtrusive,*
(xii) *universally open,*
(xiii) *smooth,*
(xiv) *unramified,*
(xv) *étale,*
(xvi) *locally separated,*
(xvii) *separated,*
(xviii) *proper,*
(xix) *affine,*
(xx) *quasi-affine,*
(xxi) *finite, or*
(xxii) *quasi-finite.*

Then f (resp. Δ_f) has property P if and only if f_λ (resp. Δ_{f_λ}) has property P for all sufficiently large λ's.

If, in addition, $X_\alpha \to Y_\alpha$ is representable and a group object, then the same conclusion holds for the properties:

(xxiii) *abelian fibers, and*
(xxiv) *quasi-finite with linearly reductive fibers.*

Proof. The condition is clearly sufficient as all properties are stable under base change. We will prove that it is necessary. We can assume that $S_0 = S_\alpha = Y_\alpha$. As the properties are fppf-local on the base, we can further assume that S_0 is an affine scheme. When f_α is strongly representable, the proposition is [$\mathrm{EGA}_{\mathrm{IV}}$, Thms. 8.10.5, 11.2.6, Prop. 17.7.8] and [Ryd10, Thms. 6.4 and 6.6] (for properties (xi) and (xii)).

(ix)–(xiii): Properties (ix) surjective, (x) flat, (xi) universally subtrusive, (xii) universally open and (xiii) smooth can be checked after replacing X_α with a smooth presentation.

(i)–(viii), (xiv)–(xvi): Assume that the proposition has been proven when f_α is representable (resp. strongly representable). Then, for general f_α (resp. representable f_α), we note that properties: (i) representable, (ii) monomorphism, (vii) universally injective, (xiv) unramified and (xvi) locally separated; correspond respectively to the properties: (ii) monomorphism, (iii) isomorphism, (ix) surjective, (xv) étale and (iv) immersion; of the diagonal which is representable (resp. strongly representable). A monomorphism is strongly representable by [Knu71, Thm. 6.15] and hence properties (iii)–(vi) follows from (ii) and the strongly representable case. Property (xv) is the conjunction of (x) and (xiv). Likewise, property (viii) is the conjunction of properties (vii), (ix) and (xii).

(xvii) and (xviii): Assume that the proposition has been proven when f_α is representable (resp. strongly representable). Then, for general f_α (resp. representable f_α) the proposition holds for property (xvii) separated, by considering the diagonal. Let f be proper. We can then assume that f_α is separated. Writing S_α as a limit $\varprojlim_\beta S_{\alpha\beta}$ of noetherian affine schemes, we can, by Proposition (B.2), assume that f_α is the pull-back of a finitely presented separated morphism $X_{\alpha\beta} \to S_{\alpha\beta}$. Then, by Chow's lemma [Knu71, Ch. IV, Thm. 3.1] and [Ols05], there exists a *scheme* $Z_{\alpha\beta}$ and a proper surjective morphism $Z_{\alpha\beta} \to X_{\alpha\beta}$. Let $Z_\alpha = Z_{\alpha\beta} \times_{S_{\alpha\beta}} S_\alpha$. It is enough to show that $Z_\lambda = Z_\alpha \times_{S_\alpha} S_\lambda \to S_\lambda$ is proper for sufficiently large λ. Thus, property (xviii) proper, follows from the strongly representable case.

(xix)–(xxi): If f is affine (resp. quasi-affine), then f factors as a closed immersion (resp. immersion) $X \hookrightarrow \mathbb{A}^n_S \to S$. As $\mathbb{A}^n_S = \mathbb{A}^n_{S_0} \times_{S_0} S$, it follows, by Proposition (B.2), that for sufficiently large λ there is a factorization $X_\lambda \to \mathbb{A}^n_{S_\lambda} \to S_\lambda$ such that $X \hookrightarrow \mathbb{A}^n_S$ is a pull-back of $X_\lambda \to \mathbb{A}^n_{S_\lambda}$. For sufficiently large λ, the latter morphism is a closed immersion (resp. immersion). If f is finite, then f_λ is affine for sufficiently large λ and we can apply the strongly representable case.

(xxii)–(xxiv): These properties can be checked on fibers and are constructible by Propositions (A.3) and (A.4). The result for these properties thus follows from [$\mathrm{EGA}_{\mathrm{IV}}$, Prop. 9.3.3]. □

Remark **(B.4).** *(cf.* [$\mathrm{EGA}_{\mathrm{IV}}$, Cor. 8.2.11]*)* — Let $X = \varprojlim_\lambda X_\lambda$ be a limit of quasi-compact and quasi-separated algebraic stacks and let $U \subseteq X$ be an open quasi-compact substack. Then, there exists an index λ and an open quasi-compact substack $U_\lambda \subseteq X_\lambda$ such that $U = U_\lambda \times_{X_\lambda} X$. This follows from the previous propositions since an open quasi-compact immersion is of finite presentation.

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