# ABSOLUTE NOETHERIAN APPROXIMATION OF ALGEBRAIC STACKS

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ABSTRACT. We show that every quasi-compact and quasi-separated algebraic stack can be approximated by a noetherian algebraic stack. We give several applications such as eliminating noetherian hypotheses in the theory of good moduli spaces.

## INTRODUCTION

The purpose of this paper is to prove the following ultimate version of absolute noetherian approximation for algebraic stacks, confirming [Ryd16, Conj. B].

**Theorem A.** *Let $X$ be an algebraic stack. The following are equivalent*

(i) *$X$ is quasi-compact and quasi-separated.*
(ii) *There exists an algebraic stack $X_0$ of finite presentation over* $\operatorname{Spec}\mathbb{Z}$ *and an affine morphism $X \to X_0$.*

When (ii) holds, we say that $X$ can be approximated and $X_0$ is an approximation of $X$ [Ryd15, Def. 7.1]. Theorem A was known for schemes, algebraic spaces, Deligne–Mumford stacks and algebraic stacks with finite stabilizers, see [Ryd15] and the references therein.

From basic results on stacks with approximation, we obtain two corollaries. Firstly, we settle [Ryd16, Conj. A].

**Corollary B** (Completeness property)**.** *Let $X$ be a quasi-compact and quasi-separated algebraic stack. Then every quasi-coherent $\mathcal{O}_X$-module is a directed colimit of $\mathcal{O}_X$-modules of finite presentation. In particular, every quasi-coherent $\mathcal{O}_X$-module of finite type is a quotient of an $\mathcal{O}_X$-module of finite presentation.*

*Proof.* Let $h\colon X \to X_0$ be an approximation. Since $X_0$ is noetherian, every quasi-coherent $\mathcal{O}_X$-module is the union of its coherent subsheaves [LMB00, Prop. 15.4]. The result then follows from considering the counit of the adjunction $(h^*, h_*)$, see [Ryd15, Prop. 4.6]. □

Since the $\mathcal{O}_X$-modules of finite presentation are exactly the compact objects, Corollary B says that $\mathbf{QCoh}(\mathcal{O}_X)$ is compactly generated, or in the terminology of [Ryd15], that $X$ is pseudo-noetherian. Secondly, we have the following stronger version of Theorem A.

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**Corollary C.** *Let $f\colon X \to Y$ be a morphism between quasi-compact and quasi-separated algebraic stacks. Then $f$ has an approximation, that is:*

(i) *there exists a finitely presented morphism $X_0 \to Y$ and an affine $Y$-morphism $X \to X_0$; and*
(ii) *there exists an inverse system $X_\lambda \to Y$ of finitely presented morphism with affine transition maps and inverse limit $X \to Y$.*

*Moreover, if $f$ is of finite type, then in (i), $X \to X_0$ can be taken to be a closed immersion and in (ii), the system can be chosen such that the transition maps are closed immersions.*

*Proof.* By the main theorem, $X$ and $Y$ can be approximated and are thus of strict approximation type [Ryd15, Def. 2.9]. It follows that $f$ also is of strict approximation type [Ryd15, Prop. 2.10 (vii), (viii)] so we can factor $f$ as $X \to X_0 \to Y$ [Ryd15, Prop. 4.8]. The remainder is now straightforward, and is spelled out in [Ryd15, Prop. 7.3, 7.4]. □

Corollary C also implies that [Ryd15, Thm. D] applies to any morphism between quasi-compact and quasi-separated algebraic stacks. Another easy consequence is that every quasi-separated, but not necessarily quasi-compact, morphism of algebraic stacks is *locally of approximation type* [HR15, §1].

**Outline of the proof of the main theorem.** Let $X$ be a quasi-compact and quasi-separated algebraic stack. Let $U \to X$ be a smooth presentation with $U = \operatorname{Spec} B$ an affine scheme. Then we can write $B$ as the union of its subalgebras $B_\lambda$ of finite type over $\mathbb{Z}$ and hence $U = \varprojlim U_\lambda$ where $U_\lambda = \operatorname{Spec}(B_\lambda)$. Unfortunately, the inverse system $U_\lambda$ does not "descend" to an inverse system $X_\lambda$ with limit $X$, not even for $\lambda$ large enough.

We solve this problem as follows. Fix a directed set $\Lambda$. In Section 1 we introduce colimits and limits of *almost shape* $\Lambda$. These are diagrams $\{X_\lambda\}_{\lambda\geq\alpha}$ in some category for some $\alpha \in \Lambda$. In Section 2 we introduce the 2-category $\mathbf{App}_\Lambda(X)$ of approximations of $X$, that is, diagrams $\{X_\lambda\}_{\lambda\geq\alpha}$ of almost shape $\Lambda$ with $X_\lambda \to \operatorname{Spec}\mathbb{Z}$ of finite presentation, affine transition maps and limit $X$. We show that this 2-category is equivalent to a *partially ordered set*. In Section 3 we show that $\mathbf{App}_\Lambda$ satisfies faithfully flat descent: given a faithfully flat morphism $X' \to X$, we have an equalizer:

$$\mathbf{App}_\Lambda(X) \longrightarrow \mathbf{App}_\Lambda(X') \rightrightarrows \mathbf{App}_\Lambda(X' \times_X X').$$

The problem alluded to above is that if we start with an arbitrary approximation of $X'$, it does not descend to $X$.

In Section 4, we show that $f^*\colon \mathbf{App}_\Lambda(Y) \to \mathbf{App}_\Lambda(X)$ admits a *right* adjoint $f_*$ if $f\colon X \to Y$ is smooth with *geometrically connected fibers*. The existence of this adjoint can be checked smooth-locally on $Y$ via descent. In this way, we can assume that $Y$ already has an approximation $Y \to Y_0$ and in this case one can write down an explicit formula for the adjoint.

The main theorem now follows from taking an arbitrary smooth presentation $U \to X$ and considering its canonical factorization $U \to \pi_0(U/X) \to X$ ("smooth dévissage"). The first morphism is smooth with connected fibers and handled by the adjoint described above. The second morphism is étale and was dealt with in [Ryd15] using étale dévissage.

The proof of the main theorem in [Ryd16] followed a similar smooth dévissage strategy. For a smooth morphism $f\colon X \to Y$ with geometrically connected fibers and a quasi-coherent sheaf $\mathcal{F}$ on $Y$, the functor $f^*\colon \mathbf{Sub}(\mathcal{F}) \to \mathbf{Sub}(f^*\mathcal{F})$, on quasi-coherent subsheaves, admits a left adjoint $f_!$. This left adjoint, extended to subalgebras, is also used in Section 4.

**Further applications.** In Section 5 we use the main theorem to eliminate noetherian or finite presentation assumptions in the following results:

(i) Algebraicity of Quot schemes and Hom-stacks [HR15, HR19].
(ii) Zariski's main theorem for stacks.
(iii) Proper coverings of separated stacks [Ols05].
(iv) Local structure and other results on good moduli spaces [AHR19].

The main theorem has also already been used in [AHHLR22, Thm. 5.1] to obtain a general local structure theorem for algebraic stacks at points with linearly reductive stabilizers without finiteness hypotheses.

## 1. Categories of limit diagrams

Let $\Lambda$ be a directed set and let $\mathbf{C}$ be a 2-category. In this section we introduce the 2-category $\operatorname{colim}'(\Lambda, \mathbf{C})$ of colimit diagrams of $\mathbf{C}$ of "almost shape $\Lambda$". Roughly speaking, an object of $\operatorname{colim}'(\Lambda, \mathbf{C})$ is a colimit diagram of shape $\Lambda_{\geq\alpha}$ for some $\alpha$ and two objects are equal if the colimit diagrams agree after increasing $\alpha$.

We let $\Lambda^{\triangleright} = \Lambda \cup \{\infty\}$ where $\infty$ is strictly larger then all elements of $\Lambda$. We say that $\overline{p}\colon \Lambda^{\triangleright} \to \mathbf{C}$ is a colimit diagram of shape $\Lambda$ if it exhibits $\overline{p}(\infty)$ as a colimit of the diagram $p = \overline{p}|_{\Lambda}$. Similarly, we consider limit diagrams of shape $\Lambda^{\mathrm{op}}$ and note that $\infty \in (\Lambda^{\triangleright})^{\mathrm{op}}$ is now the initial element.

We let $\operatorname{colim}(\Lambda, \mathbf{C}) \subset \operatorname{Fun}(\Lambda^{\triangleright}, \mathbf{C})$ denote the full 2-subcategory of colimit diagrams of shape $\Lambda$ and similarly for $\lim(\Lambda^{\mathrm{op}}, \mathbf{C}) \subset \operatorname{Fun}((\Lambda^{\triangleright})^{\mathrm{op}}, \mathbf{C})$. For any indices $\alpha \leq \beta$ in $\Lambda$, we have a restriction functor $\operatorname{colim}(\Lambda_{\geq\alpha}, \mathbf{C}) \to \operatorname{colim}(\Lambda_{\geq\beta}, \mathbf{C})$ and evaluating in $\infty$ gives us functors $\mathrm{ev}_{\infty}\colon \operatorname{colim}(\Lambda_{\geq\alpha}, \mathbf{C}) \to \mathbf{C}$. We consider the following 2-categories of (co)limit diagrams of "almost shape $\Lambda$"

$$\operatorname{colim}'(\Lambda, \mathbf{C}) = \operatorname{colim}_{\alpha\in\Lambda} \operatorname{colim}(\Lambda_{\geq\alpha}, \mathbf{C})$$
$$\lim{}'(\Lambda^{\mathrm{op}}, \mathbf{C}) = \operatorname{colim}_{\alpha\in\Lambda} \lim((\Lambda_{\geq\alpha})^{\mathrm{op}}, \mathbf{C})$$

that also come with evaluation functors $\mathrm{ev}_{\infty}$ to $\mathbf{C}$. Let us explicitly spell out the 2-category $\operatorname{colim}'(\Lambda, \mathbf{C})$.

A *strict* object of $\operatorname{colim}'(\Lambda, \mathbf{C})$ consists of an index $\alpha \in \Lambda$, objects $A_\lambda \in \mathbf{C}$ for every $\lambda \geq \alpha$ and 1-morphisms $\varphi_{\lambda_1\lambda_2}\colon A_{\lambda_1} \to A_{\lambda_2}$ for every $\lambda_2 \geq \lambda_1 \geq \alpha$ such that $\varphi_{\lambda\lambda} = \mathrm{id}_{A_\lambda}$ for every $\lambda \geq \alpha$ and $\varphi_{\lambda_2\lambda_3}\varphi_{\lambda_1\lambda_2} = \varphi_{\lambda_1\lambda_3}$ for every $\lambda_3 \geq \lambda_2 \geq \lambda_1 \geq \alpha$ and $A_\infty = \operatorname{colim}_{\alpha\leq\lambda<\infty} A_\lambda$. We denote such an object by $\{A_\lambda\}_{\lambda\geq\alpha}$ suppressing the $\varphi$'s.

A *non-strict* object is similar but the two conditions on $\varphi$ are replaced with specified 2-isomorphisms $\eta_\lambda\colon \mathrm{id}_{A_\lambda} \Rightarrow \varphi_{\lambda\lambda}$ and $\mu_{\lambda_1\lambda_2\lambda_3}\colon \varphi_{\lambda_2\lambda_3}\varphi_{\lambda_1\lambda_2} \Rightarrow \varphi_{\lambda_1\lambda_3}$ such that $\mu_{\lambda_1\lambda_2\lambda_2}(\eta_{\lambda_2} \star \mathrm{id}_{\varphi_{\lambda_1\lambda_2}}) = \mathrm{id}_{\varphi_{\lambda_1\lambda_2}} = \mu_{\lambda_1\lambda_1\lambda_2}(\mathrm{id}_{\varphi_{\lambda_1\lambda_2}} \star \eta_{\lambda_1})$ and $\mu_{\lambda_1\lambda_3\lambda_4}(\mathrm{id}_{\varphi_{\lambda_3\lambda_4}} \star \mu_{\lambda_1\lambda_2\lambda_3}) = \mu_{\lambda_1\lambda_2\lambda_4}(\mu_{\lambda_2\lambda_3\lambda_4} \star \mathrm{id}_{\varphi_{\lambda_1\lambda_2}})$ for every $\lambda_4 \geq \lambda_3 \geq \lambda_2 \geq \lambda_1 \geq \alpha$.

A 1-morphism $\{A_\lambda\}_{\lambda\geq\alpha} \to \{B_\lambda\}_{\lambda\geq\beta}$ consists of an index $\gamma\in\Lambda$, greater than $\alpha$ and $\beta$, together with 1-morphisms $f_\lambda\colon A_\lambda\to B_\lambda$ for every $\lambda\geq\gamma$ and 2-isomorphisms $\tau_{\lambda_1\lambda_2}\colon \varphi_{\lambda_1\lambda_2}f_{\lambda_1}\Rightarrow f_{\lambda_2}\varphi_{\lambda_1\lambda_2}$ such that $(\mathrm{id}_{f_{\lambda_3}}\star\mu_{\lambda_1\lambda_2\lambda_3})(\tau_{\lambda_2\lambda_3}\star\tau_{\lambda_1\lambda_2}) = \tau_{\lambda_1\lambda_3}(\mu_{\lambda_1\lambda_2\lambda_3}\star\mathrm{id}_{f_{\lambda_1}})$ for every $\lambda_3\geq\lambda_2\geq\lambda_1\geq\gamma$. We denote such an object by $\{f_\lambda\}_{\lambda\geq\gamma}$, suppressing the $\tau$'s.

A 2-morphism $\{f_\lambda\}_{\lambda\geq\gamma}\Rightarrow\{g_\lambda\}_{\lambda\geq\delta}$ is an equivalence class of the set consisting of an index $\epsilon\in\Lambda$ greater than $\gamma$ and $\delta$, together with 2-morphisms $\rho_\lambda\colon f_\lambda\Rightarrow g_\lambda$ for every $\lambda\geq\epsilon$ such that $\tau_{\lambda_1\lambda_2}(\mathrm{id}_{\varphi_{\lambda_1\lambda_2}}\star\rho_{\lambda_1}) = (\rho_{\lambda_2}\star\varphi_{\lambda_1\lambda_2})\tau_{\lambda_1\lambda_2}$ for every $\epsilon\leq\lambda_1\leq\lambda_2$. We denote such an element by $\{\rho_\lambda\}_{\lambda\geq\epsilon}$ and two elements are equivalent if they agree after increasing $\epsilon$.

Note that $\alpha,\beta,\gamma,\delta,\epsilon\in\Lambda$ whereas $\lambda,\lambda_1,\lambda_2,\lambda_3\in\Lambda^{\triangleright}$.

When $\mathbf{C}$ is the 2-category of categories, or the 2-category of categories fibered over a fixed category, then the usual coherence results show that $\mathrm{colim}'(\Lambda,\mathbf{C})$ is equivalent to the full 2-subcategory of strict objects.

## 2. The category of approximations

Let $\Lambda$ be a directed set and let $\mathbf{Stk}$ denote the 2-category of algebraic stacks. In this section, we study the category of limit diagrams of algebraic stacks of almost shape $\Lambda$ with the following additional assumptions:

**Definition (2.1).** The *category of approximations* $\mathbf{App}_\Lambda$ of almost shape $\Lambda$ is the full 2-subcategory of $\lim'(\Lambda^{\mathrm{op}},\mathbf{Stk})$ consisting of objects $\{X_\lambda\}_{\lambda\geq\alpha}$ such that

(i) $X_\lambda$ is of finite presentation over $\operatorname{Spec}\mathbb{Z}$ for all $\lambda\in\Lambda_{\geq\alpha}$, and
(ii) $\varphi_{\lambda\mu}\colon X_\mu\to X_\lambda$ is affine and schematically dominant for all $\mu\geq\lambda\geq\alpha$.

Note that $\mathbf{App}_\Lambda$ is a $(2,1)$-category since $\mathbf{Stk}$ is a $(2,1)$-category. The affine morphism $\varphi_{\lambda\mu}\colon X_\mu\to X_\lambda$ corresponds to a homomorphism $A_\lambda\to A_\mu$ of $\mathcal{O}_{X_\alpha}$-algebras and that $\varphi_{\lambda\mu}$ is schematically dominant means that $A_\lambda\to A_\mu$ is injective. All the $X_\lambda$, including $X_\infty$, are quasi-compact and quasi-separated so it makes no difference if we consider limits in $\mathbf{Stk}$ or its full 2-subcategory of quasi-compact and quasi-separated stacks.

**Example (2.2).** Let $X$ be an algebraic stack that can be approximated, that is, there exists an algebraic stack $X_0$ of finite presentation over $\operatorname{Spec}\mathbb{Z}$ and an affine morphism $h\colon X\to X_0$. Replacing $X_0$ by its schematic image, we may assume that $h$ is schematically dominant. Let $A_\infty = h_*\mathcal{O}_X$. Let $\Lambda$ be the directed set of $\mathcal{O}_{X_0}$-subalgebras $A_\lambda\subseteq A_\infty$ of finite type. Then $A_\infty = \varinjlim_\lambda A_\lambda$ since $X_0$ is noetherian [LMB00, Prop. 15.4]. If we let $X_\lambda = \operatorname{Spec}_{X_0}(A_\lambda)$, then $\{X_\lambda\}$ is an element of $\mathbf{App}_\Lambda$ with $X_\infty = X$.

The 2-category $\mathbf{App}_\Lambda$ is equivalent to the following 2-category where the intermediate $X_\lambda$, $\alpha<\lambda<\infty$, are replaced with subalgebras.

An object $\{X_\lambda\}_{\lambda\geq\alpha}$ is an algebraic stack $X_\alpha$ of finite presentation over $\operatorname{Spec}\mathbb{Z}$ together with an affine schematically dominant morphism $\varphi_\alpha\colon X_\infty\to X_\alpha$ and a colimit diagram of finitely generated subalgebras $A_\lambda\subseteq A_\infty = (\varphi_\alpha)_*\mathcal{O}_{X_\infty}$ of shape $\Lambda_{\geq\alpha}$. Here $A_\lambda = (\varphi_{\alpha\lambda})_*\mathcal{O}_{X_\lambda}$ and we do not need to specify any 2-isomorphisms analogous to $\eta_\lambda$ and $\mu_{\lambda_1\lambda_2\lambda_3}$.

A 1-morphism $\{f_\lambda\}_{\lambda\geq\gamma}\colon \{X_\lambda\}_{\lambda\geq\alpha} \to \{Y_\lambda\}_{\lambda\geq\beta}$ is a 2-commutative diagram

$$\begin{array}{ccc} X_\infty & \xrightarrow{f_\infty} & Y_\infty \\ {\scriptstyle\varphi_\gamma}\downarrow & \swarrow\!\!\!\!/\,{\scriptstyle\tau_{\gamma\infty}} & \downarrow{\scriptstyle\varphi_\gamma} \\ X_\gamma & \xrightarrow[f_\gamma]{} & Y_\gamma \end{array}$$

such that if $B_\lambda \subseteq B_\infty$ and $A_\lambda \subseteq A_\infty$ are the algebras over $\mathcal{O}_{X_\gamma}$ and $\mathcal{O}_{Y_\gamma}$ corresponding to $X_\lambda$ and $Y_\lambda$, then the image of $f_\gamma^*A_\lambda \to f_\gamma^*A_\infty \to B_\infty$ is contained in $B_\lambda \subseteq B_\infty$ for every $\lambda \geq \gamma$.

A 2-morphism $\{\rho_\lambda\}_{\lambda\geq\epsilon}\colon \{f_\lambda\}_{\lambda\geq\gamma} \Rightarrow \{g_\lambda\}_{\lambda\geq\delta}$ are 2-morphisms $\rho_\infty\colon f_\infty \Rightarrow g_\infty$ and $\rho_\epsilon\colon f_\epsilon \Rightarrow g_\epsilon$ such that $\tau_{\epsilon\infty}(\mathrm{id}_{\varphi_\epsilon} \star \rho_\infty) = (\rho_\epsilon \star \mathrm{id}_{\varphi_\epsilon})\tau_{\epsilon\infty}$. Two 2-morphisms are identified if they agree after increasing $\epsilon$.

As before we have an evaluation map $\mathrm{ev}_\infty\colon \mathbf{App}_\Lambda \to \mathbf{Stk}$.

**Proposition (2.3).** *Let $\{X_\lambda\}$ and $\{Y_\lambda\}$ be objects of $\mathbf{App}_\lambda$. Then the map of groupoids $\mathrm{Map}_{\mathbf{App}_\lambda}(\{X_\lambda\},\{Y_\lambda\}) \to \mathrm{Map}_{\mathbf{Stk}}(X_\infty,Y_\infty)$ is fully faithful.*

*Proof.* Let $\{f_\lambda\}_\lambda$ and $\{g_\lambda\}_\lambda$ be two morphisms $\{X_\lambda\} \to \{Y_\lambda\}$. For both objects and morphisms, we may assume that $\lambda \geq \alpha$ for some fixed $\alpha$. We need to show that $\Phi\colon \mathrm{Map}(\{f_\lambda\}_\lambda,\{g_\lambda\}_\lambda) \to \mathrm{Map}(f_\infty,g_\infty)$ is bijective.

Let $I_\lambda = \mathrm{Isom}_{X_\lambda}(f_\lambda,g_\lambda) = X_\lambda \times_{Y_\lambda\times Y_\lambda,\Delta} Y_\lambda$. The natural map $I_\lambda \to X_\lambda$ is representable and of finite presentation. We identify 2-morphisms $\rho_\lambda\colon f_\lambda \Rightarrow g_\lambda$ with sections of $I_\lambda \to X_\lambda$. Let $\mu \geq \lambda \geq \alpha$. Since $\varphi_{\lambda\mu}\colon X_\mu \to X_\lambda$ is affine, the induced morphism $I_\mu \to I_\lambda \times_{X_\lambda} X_\mu$ is a closed immersion.

We first show that $\Phi$ is injective. Suppose we are given $\{(\rho_i)_\lambda\}\colon \{f_\lambda\} \Rightarrow \{g_\lambda\}$ for $i=1,2$ with $\lambda \geq \alpha$, such that $(\rho_1)_\infty = (\rho_2)_\infty$. Identifying $(\rho_i)_\lambda$ with sections of $I_\lambda \to X_\lambda$, we have $(\rho_1)_\alpha\varphi_{\alpha\infty} = \varphi_{\alpha\infty}(\rho_1)_\infty = \varphi_{\alpha\infty}(\rho_2)_\infty = (\rho_2)_\alpha\varphi_{\alpha\infty}$. The two sections $(\rho_i)_\alpha\varphi_{\alpha\infty}$ of $I_\alpha \times_{X_\alpha} X_\infty \to X_\infty$ thus coincide. Since $I_\alpha \to X_\alpha$ is of finite presentation, it follows that the two sections $(\rho_i)_\alpha\varphi_{\alpha\lambda}$ of $I_\alpha \times_{X_\alpha} X_\lambda \to X_\lambda$ coincide for all sufficiently large $\lambda$. Since $I_\lambda \to I_\alpha \times_{X_\alpha} X_\lambda$ is a monomorphism, it follows that $(\rho_1)_\lambda = (\rho_2)_\lambda$ for all sufficiently large $\lambda$.

We now show that $\Phi$ is surjective. Let $\rho_\infty\colon f_\infty \Rightarrow g_\infty$ be a 2-morphism and identify it with a section of $I_\infty \to X_\infty$. We have a map $\varphi_{\alpha\infty}\rho_\infty\colon X_\infty \to I_\infty \to I_\alpha$ and since $I_\alpha \to X_\alpha$ is of finite presentation, it factors through a map $\widetilde{\rho}_\epsilon\colon X_\epsilon \to I_\alpha$ for some $\epsilon \geq \alpha$. Let $\lambda \geq \epsilon$. Then we have an induced section $\rho_\lambda\colon X_\lambda \to I_\alpha \times_{X_\alpha} X_\lambda$. But $X_\infty \to X_\lambda$ is schematically dominant, $I_\lambda \to I_\alpha \times_{X_\alpha} X_\lambda$ is a closed immersion and $\varphi_{\alpha\infty}\rho_\infty$ factors through $I_\lambda$. It follows that $\rho_\lambda$ factors uniquely through $I_\lambda$. Thus, we have a 2-morphism $\{\rho_\lambda\}_{\lambda\geq\epsilon}$. □

Proposition 2.3 says that given $\{X_\lambda\}$ and $\{Y_\lambda\}$ and $f_\infty\colon X_\infty \to Y_\infty$ there exists at most one morphism $\{f_\lambda\}$ above $f_\infty$ up to unique 2-isomorphism. In particular we have:

**Corollary (2.4).** *Let $X$ be an algebraic stack. The 2-category $\mathbf{App}_\Lambda(X) := \mathrm{ev}_\infty^{-1}(X)$ is a preordered set, hence equivalent to a partially ordered set.*

If $\{X_\lambda\},\{Y_\lambda\} \in \mathbf{App}_\Lambda(X)$, then we write $\{X_\lambda\} \geq \{Y_\lambda\}$ if there exists a morphism $\{X_\lambda\} \to \{Y_\lambda\}$.

*Remark* **(2.5).** If $\mathbf{App}_\Lambda(X)$ is not empty, then we have an affine morphism $X \to X_\alpha$ with $X_\alpha$ of finite presentation over $\mathbb{Z}$, that is, $X$ has an approximation. Conversely, if $X$ has an approximation, then we can find a directed set $\Lambda$ such that $\mathbf{App}_\Lambda(X)$ is non-empty, cf. Example 2.2.

**Corollary (2.6).** *Let $X$ be an algebraic stack. Let $h\colon X \to X_0$ be an affine schematically dominant morphism to an algebraic stack $X_0$ of finite presentation over $\mathbb{Z}$. Let $A = h_*\mathcal{O}_X$. Then $\mathbf{App}_\Lambda(X)^{\mathrm{op}}$ is equivalent to the category $\mathbf{App}_\Lambda(A)$ of colimit diagrams $\{A_\lambda\}$ of finitely generated $\mathcal{O}_{X_0}$-subalgebras of $A$ of almost shape $\Lambda$ and colimit $A_\infty = A$.*

*Proof.* We have already seen that $\mathbf{App}_\Lambda(X)$ is a preordered set and a colimit diagram $\{A_\lambda\}$ gives rise to an element of $\mathbf{App}_\Lambda(X)$ by applying $\operatorname{Spec}_{X_0}(-)$. Conversely, given $\{X_\lambda\}_{\lambda\geq\alpha}$ we may factor $X_\infty = X \to X_0$ through $X_\gamma$ for some $\gamma \geq \alpha$. After increasing $\gamma$, we can arrange so that $X_\gamma \to X_0$ is affine [Ryd15, Thm. C]. Let $A_\lambda$ be the push-forward of the structure sheaf along $X_\lambda \to X_\gamma \to X_0$ for every $\lambda \geq \gamma$. Then $\{A_\lambda\}_{\lambda\geq\gamma}$ is an object of $\mathbf{App}_\Lambda(A)$. □

If $\{A_\lambda\}, \{B_\lambda\} \in \mathbf{App}_\Lambda(A)$, then we write $\{A_\lambda\} \leq \{B_\lambda\}$ if there exists a morphism $\{A_\lambda\} \to \{B_\lambda\}$, or equivalently, if $A_\lambda \subseteq B_\lambda$ for all sufficiently large $\lambda$. Note that the convention is opposite to that in $\mathbf{App}_\Lambda(X)$.

*Remark* **(2.7).** Two elements $\{A_\lambda\}_{\lambda\geq\alpha}$ and $\{B_\lambda\}_{\lambda\geq\beta}$ have least upper bound $\{A_\lambda \cup B_\lambda\}_{\lambda\geq\gamma}$ where $\gamma$ is some upper bound of $\alpha$ and $\beta$. In particular, if $\mathbf{App}_\Lambda(X)$ is non-empty, then $\mathbf{App}_\Lambda(X)$ is an upper semi-lattice. However, $\mathbf{App}_\Lambda(X)$ is not necessarily a lattice. Even if $X_0 = \operatorname{Spec} k$ is the spectrum of a field and $A$ is a $k$-algebra, the intersection of two finitely generated sub-algebras of $A$ need not be finitely generated.

If $X$ is of finite presentation over $\operatorname{Spec}\mathbb{Z}$, then $\mathbf{App}_\Lambda(X)$ is the singleton set for any directed set $\Lambda$. If $X$ is not quasi-compact and quasi-separated, then $\mathbf{App}_\Lambda(X)$ is empty.

## 3. The stack of approximations

We have seen that the fibers of $\mathrm{ev}_\infty\colon \mathbf{App}_\Lambda \to \mathbf{Stk}$ are preordered sets. In this section, we show that $\mathbf{App}_\Lambda \to \mathbf{Stk}$ is a stack after restricting to flat and finitely presented morphisms.

### 3.1. Fibered category.

We begin by studying cartesian arrows in $\mathbf{App}_\Lambda$.

**Definition (3.1).** We say that a 1-morphism $\{f_\lambda\}\colon \{X_\lambda\} \to \{Y_\lambda\}$ in $\mathbf{App}_\Lambda$ is *cartesian* if there exists an index $\alpha \in \Lambda$ such that for every $\mu \geq \lambda \geq \alpha$, the square

$$\begin{array}{ccc} X_\mu & \xrightarrow{f_\mu} & Y_\mu \\ \downarrow{\scriptstyle \varphi_{\lambda\mu}} & & \downarrow{\scriptstyle \varphi_{\lambda\mu}} \\ X_\lambda & \xrightarrow{f_\lambda} & Y_\lambda \end{array}$$

is cartesian.

The cartesian 1-morphisms are cartesian in the sense of fibered categories:

**Proposition (3.2).** *Let* $\{h_\lambda\}\colon \{X_\lambda\} \to \{Z_\lambda\}$ *and* $\{g_\lambda\}\colon \{Y_\lambda\} \to \{Z_\lambda\}$ *be* $1$*-morphisms in* $\mathbf{App}_\Lambda$ *and suppose that we are given a commutative diagram*

$$\begin{array}{ccccc} & & Y_\infty & & \\ & \overset{f}{\nearrow} & \Downarrow \rho & \overset{g_\infty}{\searrow} & \\ X_\infty & & \xrightarrow[h_\infty]{\hspace{3em}} & & Z_\infty. \end{array}$$

*If* $\{g_\lambda\}$ *is cartesian, then there exists a* $1$*-morphism* $\{f_\lambda\}\colon \{X_\lambda\} \to \{Y_\lambda\}$ *and a* $2$*-isomorphism* $\{\rho_\lambda\}\colon \{g_\lambda f_\lambda\} \Rightarrow \{h_\lambda\}$ *such that* $f_\infty = f$ *and* $\rho_\infty = \rho$. *Moreover,* $(f_\lambda, \rho_\lambda)$ *is unique up to unique* $2$*-isomorphism.*

*Proof.* The uniqueness is Proposition 2.3. For the existence, pick an index $\alpha$ for which $\{Y_\lambda\}$ is defined. The composition $X_\infty \to Y_\infty \to Y_\alpha$ then factors through $X_\gamma \to Y_\alpha$ for some $\gamma$ that we can take to be larger than $\alpha$. After increasing $\gamma$, the two compositions $X_\gamma \to Y_\alpha \to Z_\alpha$ and $X_\gamma \to Z_\gamma \to Z_\alpha$ coincide up to some 2-isomorphism compatible with $\rho$. Since $\{g_\lambda\}$ is cartesian, this gives, for every $\lambda \geq \gamma$, a map $f_\lambda\colon X_\lambda \to Y_\lambda = Y_\alpha \times_{Z_\alpha} Z_\lambda$ together with a 2-isomorphism $\rho_\lambda\colon g_\lambda f_\lambda \Rightarrow h_\lambda$. ☐

We also have lots of cartesian arrows:

**Proposition (3.3).** *Let* $\{Y_\lambda\}$ *be an object in* $\mathbf{App}_\Lambda$. *If* $f\colon X \to Y_\infty$ *is a flat morphism of finite presentation, then there exists a cartesian arrow* $\{f_\lambda\}\colon \{X_\lambda\} \to \{Y_\lambda\}$ *such that* $X_\infty = X$ *and* $f_\infty = f$ *and* $(\{X_\lambda\}, \{f_\lambda\})$ *is unique up to unique* $2$*-isomorphism.*

*Proof.* The uniqueness is Proposition 2.3. For the existence, we note that $f\colon X_\infty \to Y_\infty$ descends to a flat and finitely presented morphism $f_\alpha\colon X_\alpha \to Y_\alpha$ for some $\alpha$ [Ryd15, App. B]. For every $\lambda \in \Lambda_{\geq\alpha}$, we let $X_\lambda = X_\alpha \times_{Y_\alpha} Y_\lambda$ and let $f_\lambda\colon X_\lambda \to Y_\lambda$ be the projection. Note that $X_\mu \to X_\lambda$ is schematically dominant since $f_\alpha$ is flat. ☐

This means that the restriction $\mathrm{ev}_\infty\colon \mathbf{App}_\Lambda^{\mathrm{flat}} \to \mathbf{Stk}^{\mathrm{flat}}$ is a fibered 2-category where $\mathbf{Stk}^{\mathrm{flat}}$ is the non-full 2-subcategory of all algebraic stacks but with 1-morphisms that are flat and of finite presentation, and $\mathbf{App}_\Lambda^{\mathrm{flat}}$ is the non-full 2-subcategory with all objects and 1-morphisms $\{f_\lambda\}$ such that $f_\infty$ is flat and of finite presentation. Equivalently, we have a 2-functor

$$\mathbf{App}_\Lambda\colon (\mathbf{Stk}^{\mathrm{flat}})^{\mathrm{op}} \to \mathbf{Pos}, \quad X \mapsto \mathbf{App}_\Lambda(X)$$

where **Pos** denotes the 2-category of partially ordered sets.

3.2. **Descent.** We will now show that $\mathbf{App}_\Lambda$ has faithfully flat descent.

**Theorem (3.4).** *Let* $f\colon X' \to X$ *be a faithfully flat morphism of finite presentation. Then* $\mathbf{App}_\Lambda$ *is a stack with respect to* $f$*, that is,*

$$\mathbf{App}_\Lambda(X) \longrightarrow \mathbf{App}_\Lambda(X') \rightrightarrows \mathbf{App}_\Lambda(X' \times_X X')$$

*is an equalizer of partially ordered sets. Equivalently, it is an equalizer of sets such that* $\mathbf{App}_\Lambda(X) \to \mathbf{App}_\Lambda(X')$ *is order-reflecting.*

*Proof.* We first show that $\mathbf{App}_\Lambda(X) \to \mathbf{App}_\Lambda(X')$ is order-reflecting. Let $\{X_{1,\lambda}\}$ and $\{X_{2,\lambda}\}$ be objects of $\mathbf{App}_\Lambda(X)$ such that $f^*\{X_{1,\lambda}\} \leq f^*\{X_{2,\lambda}\}$. Choose an algebraic stack $X_0$ of finite presentation over $\mathbb{Z}$ and an affine schematically dominant morphism $X \to X_0$, e.g., take $X_0 = X_{1,\alpha}$. After replacing $X_0$ with a finer approximation of $X$, we can find a flat morphism $f_0 \colon X_0' \to X_0$ of finite presentation such that $X' = X_0' \times_{X_0} X$. Let $A$ and $A' = f_0^* A$ be the $\mathcal{O}_{X_0}$- and $\mathcal{O}_{X_0'}$-algebras corresponding to $X$ and $X'$. By Corollary 2.6 we can identify $\mathbf{App}_\Lambda(X)^{\mathrm{op}}$ with the category $\mathbf{App}_\Lambda(A)$ of colimit diagrams $\{A_\lambda\}$ of $\mathcal{O}_{X_0}$-subalgebras of $A$ of almost shape $\Lambda$ and similarly for $X'$. Since $f^*\{A_{1,\lambda}\} \leq f^*\{A_{2,\lambda}\}$, there exists an index $\gamma$ such that $f^*A_{1,\lambda} \subseteq f^*A_{2,\lambda}$ for all $\lambda \geq \gamma$. By flat descent, it follows that $A_{1,\lambda} \subseteq A_{2,\lambda}$ for all $\lambda \geq \gamma$. That is, $\{X_{1,\lambda}\} \leq \{X_{2,\lambda}\}$.

Since $\mathbf{App}_\Lambda(X) \to \mathbf{App}_\Lambda(X')$ is order-reflecting, it is also injective. Let $X'' = X' \times_X X'$ and let $X''' = X' \times_X X' \times_X X'$. Let $\{X'_\lambda\} \in \mathbf{App}_\Lambda(X')$ such that the two pull-backs to $\mathbf{App}_\Lambda(X'')$ are equal. It remains to show that $\{X'_\lambda\}$ comes from an object of $\mathbf{App}_\Lambda(X)$.

The three pull-backs of $\{X'_\lambda\}$ to $\mathbf{App}_\Lambda(X''')$ are also equal. Choose representatives $\{X''_\lambda\}$ and $\{X'''_\lambda\}$ of the pull-backs in $\mathbf{App}_\Lambda$. Then we have flat cartesian maps

$$\{X'_\lambda\} \overset{\pi_1}{\Longleftarrow} \{X''_\lambda\} \overset{\pi_{12}}{\Lleftarrow} \{X'''_\lambda\}$$

in $\mathbf{App}_\Lambda$ and $\{X'''_\lambda\} = \{X''_\lambda\} \times_{\pi_1, \{X'_\lambda\}, \pi_2} \{X''_\lambda\}$. By Proposition 3.2, the diagonal $X' \to X' \times_X X'$ induces a map $\Delta \colon \{X'_\lambda\} \to \{X''_\lambda\}$. The maps $s = \pi_1$, $t = \pi_2$, $c = \pi_{13}$, $e = \Delta$ endows $\{X''_\lambda\} \rightrightarrows \{X'_\lambda\}$ with the structure of a groupoid in $\mathbf{App}_\Lambda$. The axioms, which involves 2-isomorphisms between various compositions and identities between 2-isomorphisms and hold for $X'' \rightrightarrows X'$, are satisfied by Proposition 2.3. Since the axioms involve a finite number of morphisms and a finite number of compositions, we may find an index $\alpha$ such that $X''_\lambda \rightrightarrows X'_\lambda$ becomes a groupoid for every $\lambda \geq \alpha$. If $X_\lambda$ denotes the stack quotient, then we obtain an element $\{X_\lambda\}_{\lambda \geq \alpha}$ of $\mathbf{App}_\Lambda(X)$ such that $f^*\{X_\lambda\} = \{X'_\lambda\}$. □

## 4. Adjoints for pure morphisms

Let $f \colon X \to Y$ be a morphism of algebraic stacks and let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_Y$-module. We obtain a functor

$$\widetilde{f^*} \colon \mathbf{Sub}(\mathcal{F}) \to \mathbf{Sub}(f^*\mathcal{F})$$

taking a quasi-coherent $\mathcal{O}_Y$-submodule $\mathcal{F}_0 \subseteq \mathcal{F}$ to the image of $f^*\mathcal{F}_0 \to f^*\mathcal{F}$. When $f$ is flat, then $\widetilde{f^*}\mathcal{F}_0 = f^*\mathcal{F}_0$. When $f$ is quasi-compact and quasi-separated, then $\widetilde{f^*}\mathcal{F}_0$ has a *right adjoint*

$$\widetilde{f_*} \colon \mathbf{Sub}(f^*\mathcal{F}) \to \mathbf{Sub}(\mathcal{F})$$

taking $\mathcal{G}_0 \subseteq f^*\mathcal{F}$ to $f_*\mathcal{G}_0 \times_{f_*f^*\mathcal{F}} \mathcal{F}$. Note that $\widetilde{f^*}$ always preserves submodules of finite type, but in general $\widetilde{f_*}$ does not. Since $\mathbf{Sub}(-)$ is a partially ordered set, $(\widetilde{f^*}, \widetilde{f_*})$ is an example of a Galois connection and $\widetilde{f_*}\mathcal{G}_0$ is the largest submodule $\mathcal{F}_0 \subseteq \mathcal{F}$ such that $\widetilde{f^*}\mathcal{F}_0 \subseteq \mathcal{G}_0$. Since $\widetilde{f^*}$ preserves unions, we also have that $\widetilde{f_*}\mathcal{G}_0$ is the union of all submodules $\mathcal{F}_i \subseteq \mathcal{F}$ such that $\widetilde{f^*}\mathcal{F}_i \subseteq \mathcal{G}_0$.

If a *left adjoint* $f_!$ to $\widetilde{f^*}$ exists, then $f_!\mathcal{G}_0$ is the intersection of all submodules $\mathcal{F}_i \subseteq \mathcal{F}$ such that $\mathcal{G}_0 \subseteq \widetilde{f^*}\mathcal{F}_i$. This intersection always makes sense, but only defines a left adjoint if $\widetilde{f^* f_! \mathcal{G}_0}$ contains $\mathcal{G}_0$. This is not always the case since $\widetilde{f^*}$ does not preserve intersections in general, even for flat $f$ [Ryd16, Ex. 6.1].

Now assume that $f$ is flat, of finite presentation, and *pure* in the sense of Raynaud–Gruson [RG71, Déf. 3.3.3], [Ryd16, Def. 4.8]. Then a left adjoint

$$f_! \colon \mathbf{Sub}(f^*\mathcal{F}) \to \mathbf{Sub}(\mathcal{F})$$

exists [Ryd16, Thm. 6.3]. It also preserves submodules of finite type and commutes with arbitrary base change $g\colon Y' \to Y$ in the sense that $\widetilde{g^*} f_! = f'_! \widetilde{g'^*}$ where $f'$ and $g'$ denote the pull-backs of $f$ along $g$ and $g$ along $f$ [Ryd16, Thm. 6.3].

This result immediately generalizes to subalgebras:

**Theorem (4.1).** *Let $f\colon X \to Y$ be flat, of finite presentation, and* pure. *Let $\mathcal{A}$ be a quasi-coherent $\mathcal{O}_Y$-algebra. Then $f^*\colon \mathbf{Sub}(\mathcal{A}) \to \mathbf{Sub}(f^*\mathcal{A})$ has a left adjoint*

$$f_!^{\mathrm{alg}} \colon \mathbf{Sub}(f^*\mathcal{A}) \to \mathbf{Sub}(\mathcal{A}).$$

*Moreover, $f_!^{\mathrm{alg}}$ preserves subalgebras of finite type and commutes with arbitrary base change.*

*Proof.* Let $\mathcal{B}_0 \subseteq f^*\mathcal{A}$ be a subalgebra. It is clear that $f_!^{\mathrm{alg}}\mathcal{B}_0$ is the smallest subalgebra containing $f_!\mathcal{B}_0 \subseteq \mathcal{A}$, that is, $f_!^{\mathrm{alg}}\mathcal{B}_0 = \operatorname{im}\big(\mathrm{Sym}_{\mathcal{O}_Y}(f_!\mathcal{B}_0) \to \mathcal{A}\big)$. Since symmetric products and images commute with pull-backs, it is also clear that $f_!^{\mathrm{alg}}$ commutes with arbitrary base change.

Now assume that $\mathcal{B}_0$ is an $\mathcal{O}_X$-algebra of finite type. To prove that $f_!^{\mathrm{alg}}\mathcal{B}_0$ is of finite type, we may work fppf-locally on $Y$ and assume that $Y$ is affine. Then $X$ is pseudo-noetherian [Ryd15, Prop. 4.8] so we may write $\mathcal{B}_0$ as the union of its $\mathcal{O}_X$-submodules of finite type. In particular, there is a submodule $\mathcal{G}_0 \subseteq \mathcal{B}_0$ of finite type such that $\mathcal{B}_0$ is the smallest subalgebra containing $\mathcal{G}_0$. Then $f_!^{\mathrm{alg}}\mathcal{B}_0$ is the smallest subalgebra containing $f_!\mathcal{G}_0$, hence of finite type. □

*Remark* **(4.2).** The right adjoint $f_*$ for submodules is also a right adjoint for subalgebras. In general, however, $f_*$ does not preserve algebras of finite type.

*Remark* **(4.3).** Let $f\colon X \to Y$ be a morphism of finite presentation between algebraic stacks. If $f$ is smooth, or more generally flat with geometrically reduced fibers, then there is a canonical factorization $X \to \pi_0(X/Y) \to Y$ where $X \to \pi_0(X/Y)$ has geometrically connected fibers and $\pi_0(X/Y) \to Y$ is étale and representable, but not necessarily separated. See [LMB00, 6.8] for $f$ smooth and representable and [Rom11, Thm. 2.5.2] for the general case. When $f$ is smooth, the morphism $X \to \pi_0(X/Y)$ is smooth with geometrically connected fibers, hence pure [RG71, Ex. 3.3.4 (iii)].

**Theorem (4.4).** *Let $f\colon X \to Y$ be a faithfully flat morphism of finite presentation between quasi-compact and quasi-separated algebraic stacks. If $f$ is smooth with geometrically connected fibers, then $f^*\colon \mathbf{App}_\Lambda(Y) \to \mathbf{App}_\Lambda(X)$*

*admits a right adjoint $f_*$. Moreover, $f_*$ commutes with base change along flat morphisms $g\colon Y' \to Y$ of finite presentation.*

*Proof.* We first prove the theorem when $Y$ has an approximation, that is, when there exists an affine schematically dominant morphism $h\colon Y \to Y_0$ with $Y_0$ of finite presentation over $\operatorname{Spec}\mathbb{Z}$. We can arrange so that $f$ descends to a smooth surjective morphism $f_0\colon X_0 \to Y_0$ of finite presentation [Ryd15, Prop. B.3]. We can also arrange so that $f_0$ has geometrically connected fibers, e.g., using that we have a factorization $X_0 \to \pi_0(X_0/Y_0) \to Y_0$ which commutes with base change (cf. Remark 4.3). Then $f_0$ is pure.

The preordered set $\mathbf{App}_\Lambda(Y)^{\mathrm{op}}$ can be identified with the category of colimit diagrams $\mathbf{App}_\Lambda(A)$ of finitely generated $\mathcal{O}_{Y_0}$-algebras of almost shape $\Lambda$ and colimit $A = h_*\mathcal{O}_Y$ (Corollary 2.6). We have a similar identification for $\mathbf{App}_\Lambda(X)$ and $f^*$ takes an object $\{A_\lambda\} \in \mathbf{App}_\Lambda(A)$ to $\{f_0^*A_\lambda\} \in \mathbf{App}_\Lambda(f_0^*A)$. I claim that $f_!\colon \mathbf{App}_\Lambda(f_0^*A) \to \mathbf{App}_\Lambda(A)$ given by $f_!(\{B_\lambda\}) = \{(f_0)_!^{\mathrm{alg}}B_\lambda\}$ is a left adjoint. This follows immediately from Theorem 4.1 except that we have to verify that $\operatorname{colim}_\lambda (f_0)_!^{\mathrm{alg}}B_\Lambda = A$. Since $(f_0)_!^{\mathrm{alg}}$ is a left adjoint, it commutes with colimits so $\operatorname{colim}_\lambda (f_0)_!^{\mathrm{alg}}B_\Lambda = (f_0)_!^{\mathrm{alg}}f_0^*A$. Since $f_0$ is faithfully flat and $f_0^*A \subseteq f_0^*(f_0)_!^{\mathrm{alg}}f_0^*A \subseteq f_0^*A$, it follows that $(f_0)_!^{\mathrm{alg}}f_0^*A = A$.

Now, drop the assumption on $Y$. Let $g\colon Y' \to Y$ be a faithfully flat morphism of finite presentation from an affine scheme $Y'$ and let $f'\colon X' \to Y'$ be the base change of $f$ and $g'\colon X' \to X$ be the base change of $g$. Then by the special case, $f'_*$ exists and commutes with flat base change on $Y'$. We may now define $f_*$ by descending $f'_*g'^*$ along $g$ (Theorem 3.4) so that $g^*f_* = f'_*g'^*$ holds by definition. Since $g^*$ and $g'^*$ are order-reflecting and $(f'^*, f'_*)$ is an adjunction, it follows that $(f^*, f_*)$ is an adjunction. □

We can now prove the main theorem of the paper. Recall that it states that an algebraic stack $X$ is quasi-compact and quasi-separated if and only if it has an approximation, or equivalently, if and only if $\mathbf{App}_\Lambda(X) \neq \emptyset$ (Remark 2.5).

*Proof of Theorem A.* If $X$ has an approximation, then it is quasi-compact and quasi-separated by definition. Conversely, if $X$ is quasi-compact and quasi-separated, let $p\colon U \to X$ be a smooth presentation with $U$ affine. Then by Remark 4.3 we have a factorization $p = gf\colon U \to X' := \pi_0(U/X) \to X$ where $f$ is smooth with geometrically connected fibers and $g$ is étale and representable. Since $U = \operatorname{Spec} A$ is affine, it can be approximated by $\operatorname{Spec}\mathbb{Z}$. If $\Lambda$ is the partially ordered set of finitely generated $\mathbb{Z}$-subalgebras of $A$, then we have a canonical element $\{U_\lambda\}_{\lambda\in\Lambda}$ of $\mathbf{App}_\Lambda(U)$. By Theorem 4.4 we have an element $f_*\{U_\lambda\} \in \mathbf{App}_\Lambda(X')$. In particular, $X'$ has an approximation. By definition, this means that $X$ is of approximation type [Ryd15, Def. 2.9] and hence also has an approximation [Ryd15, Thm. 7.10]. □

## 5. Applications

5.1. **Algebraicity of moduli spaces and stacks.** Algebraicity results for stacks with finite diagonals were obtained in [HR15, Thms. A (i) & B] without assuming locally of finite presentation. With the new approximation result we can drop the assumption on the diagonal.

**Theorem (5.1).** *Let $f\colon X \to S$ be a separated morphism of algebraic stacks.*

(i) *The Hilbert functor* $\mathrm{Hilb}_{X/S}$ *is an algebraic space, separated over $S$.*
(ii) *The stack* $\mathbf{Coh}(X/S)$ *is algebraic and has affine diagonal (relative to $S$).*
(iii) *If $\mathcal{F} \in \mathbf{QCoh}(X)$, then the functor* $\mathrm{Quot}(X/S, \mathcal{F})$ *is representable and separated over $S$.*
(iv) *The Hilbert stack $\mathscr{H}^{\mathrm{qfin}}_{X/S}$ is algebraic and has affine diagonal.*

*Proof.* By the main theorem, $f$ is locally of approximation type. The algebraicity of $\mathbf{Coh}(X/S)$, $\mathrm{Quot}(X/S, \mathcal{F})$ and $\mathrm{Hilb}_{X/S} = \mathrm{Quot}(X/S, \mathcal{O}_X)$ thus follows from [HR15, Thm. 4.4] and the algebraicity of $\mathscr{H}^{\mathrm{qfin}}_{X/S}$ follows from [HR15, Thm. 2.2]. □

Similarly, we obtain the following strengthening of [HR19, Thms. 1.2 & 1.3] on the algebraicity of Hom-stacks and Weil restrictions. Recall that a morphism $X \to Y$ has *affine stabilizers* if the following equivalent conditions hold:

(i) the diagonal $\Delta_{X/Y}$ has affine fibers;
(ii) the inertia stack $I_{X/Y}$ has affine fibers; and
(iii) for any field $k$ and point $x\colon \operatorname{Spec} k \to X$, the automorphism group scheme $\mathrm{Aut}(x)$ is affine.

**Theorem (5.2).** *Let $S$ be an algebraic stack. Let $f\colon Z \to S$ be a proper and flat morphism of finite presentation.*

(i) *If $X \to S$ is a quasi-separated morphism with affine stabilizers, then the stack*

$$\underline{\mathrm{Hom}}_S(Z, X)\colon T \mapsto \mathrm{Hom}_S(Z \times_S T, X)$$

*is algebraic and $\underline{\mathrm{Hom}}_S(Z, X) \to S$ is quasi-separated with affine stabilizers. If $X \to S$ has affine/quasi-affine/separated diagonal, then so has $\underline{\mathrm{Hom}}_S(Z, X) \to S$.*

(ii) *If $X \to Z$ is a quasi-separated morphism such that $X \to Z \to S$ has affine stabilizers, then the Weil restriction*

$$f_*X = \mathbf{R}_{Z/S}(X)\colon T \mapsto \mathrm{Hom}_Z(Z \times_S T, X)$$

*is algebraic and $f_*X \to S$ is quasi-separated with affine stabilizers. If $X \to Z$ has affine/quasi-affine/separated diagonal, then so has $f_*X \to S$.*

*Proof.* By the main theorem, $X \to S$ is locally of approximation type. The result now follows from [HR19, Cor. 9.2]. □

5.2. **Zariski's Main Theorem.** We obtain the following version of Zariski's Main Theorem, slightly generalizing [Ryd16, Thm. 8.1].

**Theorem (5.3).** *Let $X \to S$ be a representable, quasi-finite and separated morphism. If $S$ is quasi-compact and quasi-separated, then there exists a factorization $X \to X' \to S$ where $X \to X'$ is an open immersion and $X' \to S$ is finite. If in addition $X \to S$ is of finite presentation, we can arrange so that $X' \to S$ also is of finite presentation.*

*Proof.* By the main theorem, $S$ is pseudo-noetherian (Corollary B). The result is thus [Ryd15, Thm. 8.6 (ii)]. □

5.3. **Elimination of noetherian hypothesis in Chow's Lemma.** We can also remove the noetherian assumption of the main result of [Ols05]:

**Theorem (5.4).** *Let $X$ be a quasi-compact separated algebraic stack. Then there exists a proper surjective morphism $X' \to X$ where $X'$ is a separated scheme which admits an ample line bundle.*

*Proof.* Choose an approximation $X \to X_0$, that is, $X_0$ is of finite presentation over $\operatorname{Spec} \mathbb{Z}$ and $X \to X_0$ is affine. We can also assume that $X_0$ is separated [Ryd15, Thm. D]. By [Ols05] there exists a proper surjective morphism $X'_0 \to X_0$ where $X'_0$ is a quasi-projective scheme. We can now take $X' := X'_0 \times_{X_0} X$. □

5.4. **Approximation of proper morphisms.**

**Corollary (5.5).** *Let $S$ be a quasi-compact algebraic stack and let $X = \varprojlim_\lambda X_\lambda$ be an inverse limit of finitely presented $S$-stacks such that $X_\mu \to X_\lambda$ is a* closed immersion *for every $\mu \geq \lambda$. If $X \to S$ is proper, then so is $X_\lambda \to S$ for all sufficiently large $\lambda$.*

*Proof.* This follows as in the proof of [Ryd15, Cor. 6.6], replacing the use of [Ryd15, Cor. 6.5 and Thm. B] with [Ryd15, Cor. 6.7] and Theorem 5.4. □

As a consequence, we can add the property *proper* (not necessarily with finite diagonal) to the list (PC) figuring in [Ryd15, Thms. C and D].

5.5. **Applications to good moduli spaces.** Let $\mathscr{X}$ be an algebraic stack. A *good moduli space* for $\mathscr{X}$ [Alp13], [AHR19, 1.7.3] is an algebraic space $X$ together with a map $\pi\colon \mathscr{X} \to X$ such that

(i) $\pi$ is quasi-compact and quasi-separated,
(ii) $\pi_*\colon \mathbf{QCoh}(\mathscr{X}) \to \mathbf{QCoh}(X)$ is exact, also after arbitrary base change $X' \to X$, and
(iii) the unit $\mathcal{O}_X \to \pi_*\mathcal{O}_{\mathscr{X}}$ is an isomorphism.

If a good moduli space exists, then it is unique [AHR19, Thm. 3.12]. The basic examples of good moduli spaces are

(i) the GIT quotient $[\operatorname{Spec} A/G] \to \operatorname{Spec} A^G$ where $G$ is a *linearly reductive* group acting on an affine scheme $A$.
(ii) the GIT quotient $[X^{ss}(L)/G] \to X/\!\!/G$ where $G$ is a *linearly reductive* group acting on a polarized projective scheme $(X, L)$.

There is also a notion of *adequate* moduli space which is equivalent to good moduli space in characteristic zero but allows for arbitrary reductive group actions in positive characteristic. If $\pi\colon \mathscr{X} \to X$ is a good (resp. adequate) moduli space, then $\pi$ is universally closed and every fiber $\pi^{-1}(x)$ has a unique closed point $x_0$ which has linearly reductive (resp. reductive) stabilizer.

A stack $\mathscr{X}$ is *fundamental* if it is of the form $[\operatorname{Spec} A/\mathrm{GL}_n]$ for some ring $A$ and $n \in \mathbb{N}$. Then $\pi\colon \mathscr{X} \to \operatorname{Spec}(\mathscr{X}, \mathcal{O}_{\mathscr{X}}) = \operatorname{Spec} A^{\mathrm{GL}_n}$ is an adequate moduli space. A stack $\mathscr{X}$ is *linearly fundamental* if it is fundamental and $\pi$ is a good moduli space. Equivalently, $\mathscr{X}$ is fundamental (resp. linearly fundamental) if it has an affine adequate (resp. good) moduli space, affine stabilizers and the resolution property.

5.5.1. *Étale-local structure of good moduli spaces.* The following results generalize [AHR19, Thm. 6.4, Thm. 6.1, Cor. 6.11 and Prop. 6.14] by removing the assumption that $\mathscr{X}$ is of finite presentation over some algebraic space.

**Theorem (5.6).** *Let $\pi\colon \mathscr{X} \to X$ be an adequate moduli space with $X$ quasi-separated. Let $x \in |X|$ be a point. Assume that $\mathscr{X}$ has affine stabilizers and separated diagonal and that the unique closed point $x_0 \in \pi^{-1}(x)$ has linearly reductive stabilizer. Then there exists an étale neighborhood $(X', x') \to (X, x)$, with $\kappa(x') = \kappa(x)$ such that $\mathscr{X}' = \mathscr{X} \times_X X'$ is fundamental.*

*Proof.* It is enough to prove that $\mathscr{X} \times_X \operatorname{Spec} \mathcal{O}^h_{X,x}$ is fundamental since this property spreads out [AHR19, Lem. 2.15 (1)]. In particular, we may assume that $x$ is closed.

The residual gerbe $\mathscr{G}_{x_0}$ is linearly fundamental. We can thus apply the non-noetherian local structure theorem [AHHLR22, Thm. 5.1][1], to $\mathscr{W}_0 := \mathscr{X}_0 := \mathscr{G}_{x_0} \hookrightarrow \mathscr{X}$. This gives an étale morphism $f\colon \mathscr{W} \to \mathscr{X}$ such that $\mathscr{W}$ is fundamental and $f|_{\mathscr{X}_0}$ is an isomorphism. Moreover, since $\mathscr{X}$ has separated diagonal, we can arrange so that $f$ is representable [AHR19, Prop. 5.7 (2)]. Let $W$ be the adequate moduli space of $\mathscr{W}$. By Luna's fundamental lemma [AHR19, Thm. 3.14], after replacing $W$ by an open neighborhood, it holds that $\mathscr{W} := \mathscr{X} \times_X W$ and that $W \to X$ is étale. The result follows with $X' := W$. □

**Theorem (5.7)** (Local structure of good moduli spaces)**.** *Let $\pi\colon \mathscr{X} \to X$ be a good moduli space with $X$ quasi-compact and quasi-separated. Assume that $\mathscr{X}$ has affine stabilizers and separated diagonal. Then there exists a Nisnevich covering $X' \to X$ such that $\mathscr{X}' = \mathscr{X} \times_X X'$ is linearly fundamental. In particular, $\pi$ has affine diagonal.*

*Proof.* Since $\pi$ is a good moduli space, the unique closed point in every fiber has linearly reductive stabilizer. By the previous theorem, we can thus find a Nisnevich covering $X' \to X$ with $X'$ affine such that $\mathscr{X}'$ is fundamental. Since $\mathscr{X}' \to X'$ is a good moduli space, $\mathscr{X}'$ is linearly fundamental. □

**Corollary (5.8)** (Adequate with linearly reductive stabilizers is good)**.** *Let $\pi\colon \mathscr{X} \to X$ be an adequate moduli space with $X$ quasi-compact and quasi-separated. Assume that $\mathscr{X}$ has affine stabilizers and separated diagonal. Then $\pi$ is a good moduli space if and only if every closed point of $\mathscr{X}$ has linearly reductive stabilizer.*

*Proof.* It is enough to prove that $\pi$ is a good moduli space after replacing $X$ with the henselization at any closed point of $X$. We can thus assume that $X$ is local and henselian. Theorem 5.6 then tells us that $\mathscr{X}$ is fundamental and the result follows from [AHR19, Cor. 6.10]. □

**Corollary (5.9)** (Compact generation)**.** *Let $\mathscr{X}$ be a quasi-compact and quasi-separated algebraic stack with affine stabilizers and separated diagonal.*

[1] This relies on the main theorem of this paper.

*If $\mathscr{X}$ admits a good moduli space, then $\mathscr{X}$ has the Thomason condition, that is:*

(i) $\mathbf{D}_{\mathrm{qc}}(\mathscr{X})$ *is compactly generated by a countable set of perfect complexes; and*
(ii) *for every quasi-compact open substack $\mathscr{U} \subseteq \mathscr{X}$, there exists a compact perfect complex on $\mathscr{X}$ with support $\mathscr{X} \smallsetminus \mathscr{U}$.*

*Proof.* Follows exactly as in [AHR19, Prop. 6.14] using Theorem 5.7. □

5.5.2. *Approximation of stacks with good moduli spaces.* The following two results generalize [AHR19, Thm. 7.3 and Cor. 7.4] by removing the assumption that $\mathscr{X}$ has the resolution property. By slight abuse of notation, we let $\mathscr{X}$ "admits a good moduli space with affine diagonal" mean that the morphism $\pi\colon \mathscr{X} \to X$ has affine diagonal, not that the good moduli space $X$ has affine diagonal.

**Theorem (5.10).** *Let $S$ be a quasi-compact algebraic space and let $\mathscr{X} = \varprojlim_\lambda \mathscr{X}_\lambda$ be an inverse limit of quasi-compact and quasi-separated morphisms $\{\mathscr{X}_\lambda \to S\}$ of algebraic stacks with affine transition maps. Suppose that $S$ satisfies (FC) or that $\mathscr{X}$ satisfies (PC) or (N). If $\mathscr{X}$ admits a good moduli space with affine diagonal, then so does $\mathscr{X}_\lambda$ for all sufficiently large $\lambda$.*

*Proof.* The question is local on $S$ so we can assume that $S$ is quasi-separated. Before studying the system $\{\mathscr{X}_\lambda\}$, we will show that there exists an approximation of $\mathscr{X}$ over $S$ with a good moduli space.

Let $\mathscr{X} \to X$ denote the good moduli space. By Theorem 5.7, there exists an étale surjective morphism $X' \to X$ such that $\mathscr{X}' := \mathscr{X} \times_X X'$ is linearly fundamental with good moduli space $X'$. In particular, $X'$ is affine.

Write $\mathscr{X}$ as an inverse limit of stacks $\mathscr{X}_\mu \to X$ of finite presentation with affine transition maps. For sufficiently large $\mu$, the stack $\mathscr{X}_\mu \times_X X'$ is linearly fundamental [AHR19, Thm. 7.3]. Let $\mathscr{Y} := \mathscr{X}_\mu$ for one such $\mu$ so that $\mathscr{Y}' := \mathscr{Y} \times_X X'$ is linearly fundamental. Let $Y := \operatorname{Spec}_X p_*\mathcal{O}_{\mathscr{Y}}$ where $p\colon \mathscr{Y} \to X$ is the structure map. Let $Y' := Y \times_X X'$. Then $\mathscr{Y}' \to Y'$ is a good moduli space with affine diagonal. It follows that so is $\mathscr{Y} \to Y$.

Now write $Y$ as an inverse limit of stacks $Y_\alpha$ of finite presentation over $S$ with affine transition maps. For sufficiently large $\alpha$ we have an étale surjective morphism $Y'_\alpha \to Y_\alpha$ of finite presentation and a finitely presented morphism $\mathscr{Y}_\alpha \to Y_\alpha$ that pull back to $Y' \to Y$ and $\mathscr{Y} \to Y$ respectively. In particular, $\mathscr{Y}'_\alpha = \mathscr{Y}_\alpha \times_{Y_\alpha} Y'_\alpha$, gives an inverse system with limit the linearly fundamental stack $\mathscr{Y}'$. It follows that $\mathscr{Y}'_\alpha$ is linearly fundamental for sufficiently large $\alpha$ [AHR19, Thm. 7.3]. Arguing as before, we conclude that $\mathscr{Y}_\alpha \to \operatorname{Spec}_{Y_\alpha}(p_\alpha)_*\mathcal{O}_{\mathscr{Y}_\alpha}$ is a good moduli space with affine diagonal. Note that $\mathscr{X} \to \mathscr{Y} \to \mathscr{Y}_\alpha$ is affine. We have thus obtained an approximation which admits a good moduli space.

Since $\mathscr{Y}_\alpha \to S$ is of finite presentation, we obtain a map $\mathscr{X}_\lambda \to \mathscr{Y}_\alpha$ for all sufficiently large $\lambda$. After increasing $\lambda$, this map is affine [Ryd15, Thm. C]. It follows that $\mathscr{X}_\lambda$ has a good moduli space with affine diagonal. □

Theorem 5.10 says that "having a good moduli space with affine diagonal" can be included in the list of properties (PA) figuring in [Ryd15, Thms. C and D], under the assumptions (FC), (PC) or (N).

**Corollary (5.11).** *Let $\mathscr{X}$ be a quasi-compact and quasi-separated algebraic stack that admits a good moduli space with affine diagonal. Suppose that $\mathscr{X}$ satisfies (FC), (PC) or (N). Then there exists a stack $\mathscr{X}_0$ that admits a good moduli space with affine diagonal and an affine morphism $\mathscr{X} \to \mathscr{X}_0$ such that $\mathscr{X}_0$ is of finite presentation over a localization of* $\operatorname{Spec} \mathbb{Z}$.

*Proof.* If $\mathscr{X}$ satisfies (FC), let $S$ be the semi-localization of $\operatorname{Spec} \mathbb{Z}$ in the characteristics that appear in $\mathscr{X}$. Otherwise, let $S = \operatorname{Spec} \mathbb{Z}$. By the main theorem, $\mathscr{X}$ can be written as an inverse limit of finitely presented $S$-stacks with affine transition maps. The result now follows from Theorem 5.10. □

5.5.3. *Deformation of the resolution property.* The following is a variant of [AHR19, Prop. 7.8] without excellency assumptions. This implies that in [AHR19, Setup. 7.6(c)] it is enough to assume that $\mathscr{X}_0$ has the resolution property.

**Proposition (5.12).** *Let $\mathscr{X}$ be an algebraic stack with affine diagonal and affine good moduli space $X$. Let $\mathscr{X}_0 \hookrightarrow \mathscr{X}$ be a closed substack with good moduli space $X_0$, which is a closed subscheme of $X$. Suppose that $(X, X_0)$ is an affine henselian pair and that $\mathscr{X}_0$ satisfies (FC), (PC) or (N). If $\mathscr{X}_0$ has the resolution property, then so does $\mathscr{X}$.*

*Proof.* Since $(X, X_0)$ is an henselian pair, $\mathscr{X}$ also satisfies (FC), (PC) or (N) [AHR19, Rmk. 7.1]. By the main theorem, we can write $\mathscr{X}_0 \hookrightarrow \mathscr{X}$ as an inverse limit of finitely presented closed immersions $\mathscr{X}_\alpha \hookrightarrow \mathscr{X}$. For sufficiently large $\alpha$, $\mathscr{X}_\alpha$ also has the resolution property [AHR19, Lem. 2.15 (1)]. Note that $(X, X_\alpha)$ also is a henselian pair. After replacing $\mathscr{X}_0$ with $\mathscr{X}_\alpha$ we can thus assume that $\mathscr{X}_0 \hookrightarrow \mathscr{X}$ is of finite presentation.

By Corollary 5.11, we have that $\mathscr{X} = \varprojlim_\lambda \mathscr{X}_\lambda$ where $\mathscr{X}_\lambda$ is essentially of finite presentation over $\operatorname{Spec} \mathbb{Z}$ and admits a good moduli space. For sufficiently large $\lambda$ we have a closed immersion $\mathscr{X}_{\lambda,0} \hookrightarrow \mathscr{X}_\lambda$ such that $\mathscr{X}_0 = \mathscr{X}_{\lambda,0} \times_{\mathscr{X}_\lambda} \mathscr{X}$ and such that $\mathscr{X}_{\lambda,0}$ has the resolution property.

Let $(X_\lambda, X_{\lambda,0})$ be the good moduli spaces of $(\mathscr{X}_\lambda, \mathscr{X}_{\lambda,0})$. Then $X_\lambda$ is essentially of finite type over $\operatorname{Spec} \mathbb{Z}$, hence excellent. We can thus apply [AHR19, Prop. 7.8 with Setup 7.6(b)] to deduce that $\mathscr{X}_\lambda \times_{X_\lambda} X_\lambda^h$ has the resolution property where $X_\lambda^h$ is the henselization of $X_\lambda$ along $X_{\lambda,0}$. Since $X \to X_\lambda$ factors through $X_\lambda^h$, we obtain an affine morphism $\mathscr{X} \to \mathscr{X}_\lambda \times_{X_\lambda} X_\lambda^h$ and it follows that $\mathscr{X}$ has the resolution property. □

## References


[AHHLR22] Jarod Alper, Jack Hall, Daniel Halpern-Leistner, and David Rydh, *Artin algebraization for pairs with applications to the local structure of stacks and Ferrand pushouts*, To appear in *Forum Math. Sigma*, May 2022, arXiv:2205.08623.

[AHR19] Jarod Alper, Jack Hall, and David Rydh, *The étale local structure of algebraic stacks*, Preprint, Dec 2019, arXiv:1912.06162.

[Alp13] Jarod Alper, *Good moduli spaces for Artin stacks*, Ann. Inst. Fourier (Grenoble) **63** (2013), no. 6, 2349–2402.

[HR15] Jack Hall and David Rydh, *General Hilbert stacks and Quot schemes*, Michigan Math. J. **64** (2015), no. 2, 335–347.

[HR19] Jack Hall and David Rydh, *Coherent Tannaka duality and algebraicity of Hom-stacks*, Algebra Number Theory **13** (2019), no. 7, 1633–1675.

[LMB00] Gérard Laumon and Laurent Moret-Bailly, *Champs algébriques*, Springer-Verlag, Berlin, 2000.

[Ols05] Martin Olsson, *On proper coverings of Artin stacks*, Adv. Math. **198** (2005), no. 1, 93–106.

[RG71] Michel Raynaud and Laurent Gruson, *Critères de platitude et de projectivité. Techniques de "platification" d'un module*, Invent. Math. **13** (1971), no. 1, 1–89.

[Rom11] Matthieu Romagny, *Composantes connexes et irréductibles en familles*, Manuscripta Math. **136** (2011), no. 1-2, 1–32.

[Ryd15] David Rydh, *Noetherian approximation of algebraic spaces and stacks*, J. Algebra **422** (2015), 105–147.

[Ryd16] David Rydh, *Approximation of sheaves on algebraic stacks*, Int. Math. Res. Not. IMRN (2016), no. 3, 717–737.

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