FAMILIES OF ZERO CYCLES AND DIVIDED POWERS: II. THE UNIVERSAL FAMILY

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Abstract. In this paper, we continue the study of the scheme of divided powers $\Gamma^d(X/S)$. In particular, we construct the universal family of $\Gamma^d(X/S)$ as a family of cycles supported on $\Gamma^{d-1}(X/S) \times_S X$ and discuss the "Hilbert-Chow" morphism. We also give a description of the k-points of $\Gamma^d(X/S)$ as effective zero cycles with certain *rational* coefficients and give an alternative description of families of zero cycles as *multivalued* morphisms. Finally, we construct sheaves of divided powers and a generalized norm functor.

Introduction

Let X/S be a separated algebraic space. In [Ryd08a], a natural functor $\underline{\Gamma}^d_{X/S}$ from S-schemes to sets parameterizing effective zero-cycles of degree d was introduced and shown to be an algebraic space — the space of divided powers $\Gamma^d(X/S)$. This is a globalization of the algebra of divided powers and the "correct" Chow scheme of points on X/S. Indeed, the space of divided powers commutes with base change and coincides with the symmetric product $\mathrm{Sym}^d(X/S)$ in characteristic zero or when X/S is flat, e.g., when $X = \mathbb{P}^n_S$. In particular, we obtain a functorial description of $\mathrm{Sym}^d(X/S)$ in the flat case.

We let $\Gamma^d_1(X/S) = \Gamma^{d-1}(X/S) \times_S X$. A geometric point of $\Gamma^d_1(X/S)$ is a zero-cycle of degree d with one marked point. It is thus expected that the addition morphism $\Phi_{X/S} : \Gamma^d_1(X/S) \to \Gamma^d(X/S)$, which forgets the marked point, should be related to the universal family of $\Gamma^d(X/S)$. When the addition morphism $\Phi_{X/S}$ is *flat*, then it has a tautological family of cycles given by the norm. Iversen [Ive70, Thm. II.3.4] showed that if $\Phi_{X/S}$ is flat, then $\Phi_{X/S}$ together with the norm family is the universal family. It should be noted that $\Phi_{X/S}$ is rarely flat, the notable exception being when X/S is a *smooth curve*. The main result of this paper is a generalization of Iversen's result to arbitrary X/S for which $\Phi_{X/S}$ need not be flat. More precisely, we construct a family of zero cycles on $\Phi_{X/S}$, that is, a morphism $\varphi_{X/S} : \Gamma^d(X/S) \to \Gamma^d(\Gamma^d_1(X/S))$, and show that it is the universal family.

Multiplicative polynomial laws. To define the universal family, we need a couple of results on multiplicative laws. Firstly, we show in §1 that it is enough to consider the category of polynomial A-algebras in the definition

Date: 2008-04-11.

2000 *Mathematics Subject Classification.* Primary 14C05; Secondary 14C25.

Key words and phrases. Families of cycles, zero cycles, divided powers, universal family, Weil restriction, norm functor, Hilbert scheme.

of a multiplicative law $B \to C$ of A-algebras. Secondly, we define the norm law of a locally free algebra in §3. Thirdly, we construct *universal shuffle laws* in §4. These are canonical multiplicative laws $\Gamma^{d_1}_A(B) \otimes \Gamma^{d_2}_A(B) \to \Gamma^{d_1+d_2}_A(B)$ of degree $((d_1, d_2))$ for positive integers d_1 and d_2. Apparently, it is difficult to directly define these laws. It is however easy to define canonical multiplicative laws $\mathrm{TS}^{d_1}_A(B) \otimes \mathrm{TS}^{d_2}_A(B) \to \mathrm{TS}^{d_1+d_2}_A(B)$ and we use these laws to define the universal shuffle laws. The universal shuffle law with $d_1 = d-1$ and $d_2 = 1$ will be of particular interest as this law gives a description of the universal family of $\Gamma^d_A(B)$, cf. Proposition (4.8).

The universal family. From the functorial description of $\Gamma^d(X/S)$ we have that the identity on $\Gamma^d(X/S)$ corresponds to a family of cycles on X parameterized by $\Gamma^d(X/S)$ — the *universal family*. The image of the universal family is a closed subspace Z_{univ} of $\Gamma^d(X/S) \times_S X$ which is integral over $\Gamma^d(X/S)$. The nilpotent structure of this subspace is difficult to describe and we do not accomplish this. However, in §5 we show that Z_{univ} is contained in the closed subscheme $\Gamma^d_1(X/S) := \Gamma^{d-1}(X/S) \times_S X \hookrightarrow \Gamma^d(X/S) \times_S X$ which has the same underlying topological space as Z_{univ}. In fact, we construct a family of cycles on $\Gamma^d_1(X/S) \to \Gamma^d(X/S)$ and show that this induces the identity on $\Gamma^d(X/S)$. This result is a globalization of the universal shuffle law in §4 described above. When $\Gamma^d_1(X/S) \to \Gamma^d(X/S)$ is flat and generically étale then the scheme $\Gamma^d_1(X/S)$ completely determines the universal family.

Relation with the Hilbert scheme. In §6 we briefly mention the natural morphism from the Hilbert scheme of d points on X to $\Gamma^d(X/S)$. This morphism takes a flat family to its determinant law and is known as the *Grothendieck-Deligne norm map*. When $\Gamma^d_1(X/S)$ is flat and generically étale over $\Gamma^d(X/S)$, the morphism $\mathrm{Hilb}^d(X/S) \to \Gamma^d(X/S)$ is an isomorphism. In particular, it is an isomorphism over the non-degeneracy locus $\Gamma^d(X/S)_{\mathrm{nondeg}}$ and an isomorphism when X/S is a family of smooth curves.

Points of $\Gamma^d(X/S)$. In §8 we describe the k-points of $\Gamma^d(X/S)$. If k is a *perfect* field, then the k-points of $\Gamma^d(X/S)$ correspond to effective zero cycles of degree d on X_k with integral coefficients. For an arbitrary field k there is a similar correspondence if we also allow certain rational coefficients. The denominators of these coefficients are powers of the characteristic of k and the maximal exponent allowed is explicitly determined. This result also follows from [Kol96, Thm. I.4.5], using that the k-points of the space of divided powers and the Chow variety coincide, but our proof is more direct.

Multi-morphisms. Let X be a scheme such that any set of d points is contained in an affine open subset, e.g., let X be quasi-projective. There is then another striking description of families of zero cycles of degree d on X parameterized by any space T, that is, of morphisms $T \to \Gamma^d(X/S)$. We show that a family can be described as a *multi-morphism* $f : T \to X$ of degree d. This consists of a multivalued map $f : T \to X$ together with a semi-local multiplicative law $\theta : \mathcal{O}_X \to f_*\mathcal{O}_T$. The formalism is very close to that of ordinary morphisms of schemes. The condition on X is used to ensure that for every point $t \in T$ the *set* $f(t) \subseteq X$ is contained in an affine subset. Similarly, a morphism of algebraic spaces $f : T \to X$ cannot be described as a morphism of locally ringed spaces unless every point in X has an affine neighborhood, that is, unless X is a scheme.

Norm functor and Weil restriction. Let $f : X \to Y$ be a morphism. The Weil restriction $\mathbf{R}_{X/Y}$ is a functor from X-schemes to Y-schemes defined by the property $\mathrm{Hom}_Y(T, \mathbf{R}_{X/Y}(W)) = \mathrm{Hom}_X(T \times_Y X, W)$. The existence of the Weil restriction of W, under suitable conditions on f and W, can be established using Hilbert schemes [FGA, BLR90, Ryd08c]. The *norm functor* $N_{X/Y}$ is a closely related functor which can be defined not only for X-schemes but also for sheaves on X. The existence of the norm functor is shown using a space or sheaf of divided powers. The classical setting is when X/Y is flat of constant rank d and $\mathcal{L}$ is an invertible sheaf on X [$\mathrm{EGA}_{\mathrm{II}}$, §6.5]. For affine schemes and X/Y flat, the norm functor has been studied intensively by Ferrand [Fer98] and we generalize these results.

Notation and conventions. We denote a *closed* immersion of schemes or algebraic spaces with $X \hookrightarrow Y$. When A and B are rings or modules we use $A \hookrightarrow B$ for an injective homomorphism. We let $\mathbb{N}$ denote the set of non-negative integers $0, 1, 2, \ldots$ and use the notation $((a, b)) = \binom{a+b}{a}$ for binomial coefficients.

Contents

1. Determination of a multiplicative law

Recall [Rob63] that a polynomial law $F : M \to N$ is a set of maps $F_{A'} : M \otimes_A A' \to N \otimes_A A'$ for every A-algebra A' which are natural with respect to A-algebra homomorphisms. The law F is *homogeneous* of degree d if $F_{A'}(a'x') = a'^d F_{A'}(x')$ for every A-algebra A' and elements $a' \in A'$ and $x' \in M \otimes_A A'$. If M and N are A-algebras, then we say that F is multiplicative if $F_{A'}(1) = 1$ and $F_{A'}(x'y') = F_{A'}(x')F_{A'}(y')$ for every A-algebra A' and every $x', y' \in M \otimes_A A'$.

In some cases, cf. §§3–4, it is not clear that a natural map $M \to N$ extends functorially to any base change. The following proposition shows that it is enough to consider polynomial base changes.

Proposition (1.1). *Let M and N be A-modules.*

(i) *In the definition of polynomial laws we can replace the category A–**Alg** of A-algebras with the full subcategory of polynomial rings over A. To be precise, there is a one-to-one correspondence between polynomial laws $F : M \to N$ and sets of maps*

$$F_n : M[t_1, t_2, \dots, t_n] \to N[t_1, t_2, \dots, t_n], \quad n \in \mathbb{N}$$

such that $F_m \circ (\mathrm{id}_M \otimes \varphi) = (\mathrm{id}_N \otimes \varphi) \circ F_n$ for any A-algebra homomorphism $\varphi : A[t_1, t_2, \dots, t_n] \to A[t_1, t_2, \dots, t_m]$. This correspondence is given by $F \mapsto (F_{A[t_1,t_2,\dots,t_n]})_{n \in \mathbb{N}}$.

(ii) *If (F_n) is homogeneous of degree d, that is, if $F_n(az) = a^d F_n(z)$ for every $n \geq 0$, $a \in A[t_1, t_2, \dots, t_n]$ and $z \in M[t_1, t_2, \dots, t_n]$, then the corresponding polynomial law F is homogeneous of degree d.*

In particular, in the definition of (homogeneous) polynomial laws, it is enough to consider smooth A-algebras.

Proof. (i) It is immediately seen that to give a set of maps $\{F_n\}_n$ for $n \in \mathbb{N}$ commuting with A-algebra homomorphisms φ as in the proposition is equivalent to give a single map $F' : M[t_1, t_2, \dots] \to N[t_1, t_2, \dots]$ such that for every endomorphism φ of $A[t_1, t_2, \dots]$ the diagram

$$\begin{array}{ccc} M[t_1, t_2, \dots] & \xrightarrow{\mathrm{id}_M \otimes \varphi} & M[t_1, t_2, \dots] \\ \downarrow{\scriptstyle F'} & & \downarrow{\scriptstyle F'} \\ N[t_1, t_2, \dots] & \xrightarrow{\mathrm{id}_N \otimes \varphi} & N[t_1, t_2, \dots] \end{array} \tag{1.1.1}$$

commutes. A map F' such that (1.1.1) commutes, gives a unique polynomial law $F : M \to N$ such that $F' = F_{A[t_1,t_2,\dots]}$ [Rob63, Prop. IV.4, p. 271]. Moreover, if $f : \Gamma_A(M) \to N$ is the corresponding homomorphism, then $f\big(\gamma^{d_1}(x_1) \times \gamma^{d_2}(x_2) \times \cdots \times \gamma^{d_n}(x_n)\big)$ is the coefficient of $t_1^{d_1} t_2^{d_2} \dots t_n^{d_n}$ in $F'(x_1 t_1 + x_2 t_2 + \cdots + x_n t_n)$.

(ii) Let $z = x_1 t_1 + x_2 t_2 + \cdots + x_n t_n \in M[t_1, t_2, \dots, t_n]$ be a homogeneous polynomial of degree one. If F_n is homogeneous of degree d then we have that

$$t_{n+1}^d F'(z) = F'(t_{n+1} z) = (F' \circ (\mathrm{id}_M \otimes \varphi))(z) = (\mathrm{id}_N \otimes \varphi)(F'(z))$$

where φ is given by $t_i \mapsto t_{n+1} t_i$. It follows that $F'(z) \in N[t_1, t_2, \dots, t_n]$ is homogeneous of degree d and thus that $f : \Gamma_A(M) \to N$ factors through the projection $\Gamma_A(M) \to \Gamma_A^d(M)$. In particular, we have that F is homogeneous of degree d. □

Proposition (1.2). *Let B and C be A-algebras. In the correspondence between polynomial laws $F : B \to C$ and sets of maps (F_n) as in Proposition (1.1), multiplicative polynomial laws correspond to multiplicative maps, i.e., maps (F_n) such that*

(i) $F_n(1_B) = 1_C$.

(ii) $F_n(xy) = F_n(x)F_n(y), \quad \forall x, y \in B[t_1, t_2, \dots, t_n]$.

In particular, in the definition of a multiplicative polynomial law it is enough to consider smooth A-algebras.

Proof. If F is a multiplicative law, then $F_n = F_{A[t_1,t_2,\dots,t_n]}$ is multiplicative by definition. Conversely, assume that we are given a set (F_n) of multiplicative maps. This set of maps corresponds to a polynomial law $F : B \to C$ such that $F_n = F_{A[t_1,t_2,\dots,t_n]}$ by Proposition (1.1). It is clear that $F(1_B) = 1_C$. Let A' be an A-algebra and $x, y \in B \otimes_A A'$. Then there is a positive integer n, a homomorphism $A[t_1, t_2, \dots, t_n] \to A'$ and $x_n, y_n \in B[t_1, t_2, \dots, t_n]$ such that x_n and y_n are mapped to x and y respectively. The multiplicativity of F_n implies that $F_{A'}(xy) = F_{A'}(x)F_{A'}(y)$. □

2. Inhomogeneous families

It is sometimes convenient to work with families which do not have constant degree. We therefore make the following definition:

Definition (2.1). Let X/S be a separated algebraic space. We let $\Gamma^\star(X/S) = \coprod_{d\geq 0} \Gamma^d(X/S)$ and let $\underline{\Gamma}^\star_{X/S}(-) = \mathrm{Hom}_S(-, \Gamma^\star(X/S))$ be the corresponding functor.

Thus, by definition, a morphism $\alpha : T \to \Gamma^\star(X/S)$ corresponds to an open and closed partition $T = \coprod_{d\geq 0} T_d$ and families $\alpha_d : T_d \to \Gamma^d(X/S)$. We let

$$\mathrm{Image}(\alpha) = \coprod_{d\geq 0} \mathrm{Image}(\alpha_d) \hookrightarrow \coprod_{d\geq 0} X \times_S T_d = X \times_S T$$

and $\mathrm{Supp}(\alpha) = \mathrm{Image}(\alpha)_{\mathrm{red}}$. We say that the degree of α at $t \in T$ is d if $\alpha(t) \in \Gamma^d(X/S)$.

Proposition (2.2) ([Zip86, Prop. 1.7.9 a)])**.** *Let $F : B \to C$ be a multiplicative law of A-algebras. Then there is an integer n, a complete set of orthogonal idempotents $e_0, e_1, \dots, e_n$ in C and a canonical decomposition $F = F_0 + F_1 + F_2 + \cdots + F_n$ where $F_d : B \to Ce_d$ is a homogeneous multiplicative law of degree d. Note that $e_d = 0$ is possible.*

Note that conversely if $e_0, e_1, \dots, e_n$ is a complete set of orthogonal idempotents and $\big(F_d : B \to Ce_d\big)_{d=0,1,\dots,n}$ are multiplicative laws of degrees $0, 1, \dots, n$, then $F = F_0 + F_1 + \cdots + F_n$ is a multiplicative law. In fact, $F(1) = \sum_i e_i = 1$ and $F(x)F(y) = \sum_i F_i(x)F_i(y) = F(xy)$.

Theorem (2.3). *Let $S = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(B)$ be affine schemes and let $T = \mathrm{Spec}(A')$ be an affine S-scheme. Then there is a one-to-one correspondence between multiplicative laws $B \to A'$ and inhomogeneous families $T \to \Gamma^\star(X/S)$. This correspondence takes $f : T \to \Gamma^\star(X/S)$ onto $\Gamma(f) \circ (\gamma^0, \gamma^1, \gamma^2, \dots) : B \to A'$. The expression $\Gamma(f)$ is the induced map $\Gamma(\Gamma^\star(X/S)) = \prod_{d\geq 0} \Gamma^d_A(B) \to \Gamma(T) = A'$ on global sections.*

Proof. As T is quasi-compact, any morphism $f : T \to \Gamma^\star(X/S)$ factors through $\Gamma^{\leq n}(X/S) = \coprod_{d\leq n} \Gamma^d(X/S)$. The theorem thus follows from Proposition (2.2). □

3. Determinant laws and étale families

Let A be a ring, B an A-algebra and M a B-module which is free of rank d as an A-module. We then have the determinant or norm map

$$\mathrm{N}_{B/A} : B \to \mathrm{End}_A(M) \to \mathrm{End}_A(\wedge^d M) = A$$

where the first map takes b to the endomorphism on M which is multiplication by B. This map extends to a homogeneous multiplicative polynomial law which we denote the *determinant law*. We can also extend this definition to B-modules M which are locally free of rank d over A taking an open cover of $\mathrm{Spec}(A)$. Similarly, if M is locally free but not of constant rank, then we obtain an inhomogeneous multiplicative law $\mathrm{N}_{B/A} : B \to A$.

Assume now that A is an integral domain with fraction field K, that B is an A-algebra and that M is a B-module which is of finite type as an A-module but not necessarily flat. If we let d be the generic rank of M then we have the norm map

$$\mathrm{N}_{B/A} : B \to \mathrm{End}_A(M) \to \mathrm{End}_K(M \otimes_A K) \to \mathrm{End}_K\big(\wedge^d(M \otimes_A K)\big) = K$$

and according to [EGA$_{\mathrm{II}}$, Prop. 6.4.3] the elements $\mathrm{N}_{B/A}(b)$ are integral over A. In particular, if A is in addition *integrally closed* then $\mathrm{N}_{B/A}$ has image A. Under this assumption this map extends to a determinant law as it is enough to define the multiplicative polynomial law over the integrally closed polynomial rings $A[t_1, \dots, t_n]$ by Proposition (1.2).

Definition (3.1). Let S be an algebraic space and $f : X \to S$ affine. Let $\mathcal{F}$ be a quasi-coherent sheaf on X such that $f_*\mathcal{F}$ is a finite $\mathcal{O}_S$-module and one of the following conditions holds:

(i) $f_*\mathcal{F}$ is a locally free $\mathcal{O}_S$-module.
(ii) S is normal.

To $\mathcal{F}$ we associate the canonical family $\mathcal{N}_{\mathcal{F}} : S \to \Gamma^\star(X/S)$ given by the determinant law. To abbreviate, we let $\mathcal{N}_X = \mathcal{N}_{\mathcal{O}_X}$ when this is defined.

Proposition (3.2). *Let S be an algebraic space and let X/S be finite and étale. Then $\mathcal{N}_X$ is the unique morphism $S \to \Gamma^\star(X/S)$ such that $\mathrm{Supp}(\mathcal{N}_X) = X_{\mathrm{red}}$ and such that the degree of $\mathcal{N}_X$ at a point $s \in S$ is the rank of X/S at s. Furthermore we have that $\mathrm{Image}(\mathcal{N}_X) = X$. In particular, the image of $\mathcal{N}_X$ commutes with arbitrary base change.*

Proof. The question is local on S so we can assume that X/S is of constant rank d. Let $S' \to S$ be an étale cover such that $X' = X \times_S S' \to S'$ trivializes, i.e., such that $X' = S'^{\amalg d}$. It is clear that the only family $S' \to \Gamma^d(X'/S')$ with support X'_{red} is the family with multiplicity one on each component. This is given by the morphism $S' \cong \Gamma^1_{S'}(S')^{\times_{S'} d} \hookrightarrow \Gamma^d(X')$. The corresponding multiplicative law is the multiplication map $(\mathcal{O}_{S'})^d \to \mathcal{O}_{S'}$ which coincides with the determinant law. Thus $\mathcal{N}_{X'}$ is the unique family with support X'_{red}. As the image commutes with étale base change, the last statement of the proposition follows. $\square$

4. Universal shuffle laws

The A-algebra $\Gamma^d_A(B)$ represents multiplicative polynomial laws of degree d [Fer98, Prop. 2.5.1]. We thus have a canonical bijection

$$\operatorname{Hom}_{A\text{–}\mathbf{Alg}}\big(\Gamma^d_A(B), A'\big) \to \operatorname{Pol}^d_A(B, A') = \operatorname{Pol}^d_{A'}(A' \otimes_A B, A')$$

and under this correspondence, the identity on $\Gamma^d_A(B)$ corresponds to the *universal law* $U : \Gamma^d_A(B) \otimes_A B \to \Gamma^d_A(B)$. There is a natural surjection, the *canonical homomorphism* of Iversen,

$$\omega : \Gamma^d_A(B) \otimes_A B \to \Gamma^{d-1}_A(B) \otimes_A B$$

and we will show that U factors through ω. For this purpose, we first construct the multiplicative shuffle law $\mathrm{SL} : \Gamma^{d-1}_A(B) \otimes_A B \to \Gamma^d_A(B)$.

(4.1) We recall [Ryd08a, 1.2.14] that the *universal multiplication of laws*

$$\rho_{d_1,d_2} : \Gamma^{d_1+d_2}_A(M) \to \Gamma^{d_1}_A(M) \otimes_A \Gamma^{d_2}_A(M)$$

is the homomorphism corresponding to the law $x \mapsto \gamma^{d_1}(x) \otimes \gamma^{d_2}(x)$. In particular, we have that

$$\rho_{d_1,d_2}\big(\gamma^{\nu}(x)\big) = \sum_{\substack{\nu_1+\nu_2=\nu \\ |\nu_1|=d_1,\ |\nu_2|=d_2}} \gamma^{\nu_1}(x) \otimes \gamma^{\nu_2}(x). \tag{4.1.1}$$

(4.2) *The shuffle product* — For any A-module M, the product of $\Gamma_A(M)$ gives A-module homomorphisms

$$\times : \Gamma^{d_1}_A(M) \otimes_A \Gamma^{d_2}_A(M) \to \Gamma^{d_1+d_2}_A(M).$$

The composition of the universal multiplication of laws ρ_{d_1,d_2} followed by $\times$ is multiplication by $((d_1, d_2))$. In particular, if $((d_1, d_2))$ is invertible in A, then $x \otimes y \mapsto ((d_1, d_2))^{-1} x \times y$ is a retraction of ρ_{d_1,d_2}. If B is an A-algebra, then $\times$ is $\Gamma^{d_1+d_2}_A(B)$-linear.

(4.3) *The multiplicative shuffle law* — Let M be a *flat* A-module. The product on $\Gamma_A(M)$ is then identified with the *shuffle product*:

$$\times : \mathrm{TS}^{d_1}_A(M) \otimes_A \mathrm{TS}^{d_2}_A(M) \to \mathrm{TS}^{d_1+d_2}_A(M)$$

which is given by

$$x \times y = \sum_{\sigma \in \mathfrak{S}_{d_1,d_2}} \sigma(x \otimes y)$$

where the sum is taken in $\mathrm{T}^{d_1+d_2}_A(M)$. If $B = M$ is a flat A-algebra we can replace the sum with a product. This gives a multiplicative map

$$\mathrm{SL} : \mathrm{TS}^{d_1}_A(B) \otimes_A \mathrm{TS}^{d_2}_A(B) \to \mathrm{TS}^{d_1+d_2}_A(B) \tag{4.3.1}$$

defined by

$$\mathrm{SL}(z) = \prod_{\sigma \in \mathfrak{S}_{d_1,d_2}} \sigma(z).$$

Indeed, the set $\mathfrak{S}_{d_1,d_2}$ is a set of representatives of the left cosets of the subgroup $\mathfrak{S}_{d_1} \times \mathfrak{S}_{d_2} \hookrightarrow \mathfrak{S}_{d_1+d_2}$. If $z \in \mathrm{TS}^{d_1}_A(B) \otimes_A \mathrm{TS}^{d_2}_A(B)$ then $\sigma(z) = \sigma'(z)$ if σ and σ' belongs to the same left coset. As left multiplication on $\mathfrak{S}_{d_1+d_2}$ permutes the cosets, it is clear that $\mathrm{SL}(z)$ is invariant under $\mathfrak{S}_{d_1+d_2}$.

The composition of ρ_{d_1,d_2} followed by SL is taking $((d_1,d_2))^{\text{th}}$ powers and SL extends to a multiplicative law which is homogeneous of degree $((d_1,d_2))$. In fact, by Proposition (1.2) it is enough to show that SL extends functorially to

$$\mathrm{SL}_n \;:\; \mathrm{TS}^{d_1}_A(B)\otimes_A \mathrm{TS}^{d_2}_A(B)[t_1,t_2,\dots,t_n] \to \mathrm{TS}^{d_1+d_2}_A(B)[t_1,t_2,\dots,t_n]$$

which is easily seen.

Definition (4.4). Let B be a flat A-algebra. The *shuffle homomorphism* is the homomorphism

$$\Lambda^{d_1,d_2} \;:\; \Gamma^{((d_1,d_2))}_{\Gamma^{d_1+d_2}_A(B)}\big(\Gamma^{d_1}_A(B)\otimes_A \Gamma^{d_2}_A(B)\big) \to \Gamma^{d_1+d_2}_A(B)$$

which corresponds to the shuffle law constructed in (4.3).

Proposition (4.5). *Let d_1,d_2 be integers and $N=((d_1,d_2))$. The shuffle homomorphism, defined in* (4.4) *for flat A-algebras B, extends uniquely to a homomorphism*

$$\Lambda^{d_1,d_2} \;:\; \Gamma^{N}_{\Gamma^{d_1+d_2}_A(B)}\big(\Gamma^{d_1}_A(B)\otimes_A \Gamma^{d_2}_A(B)\big) \to \Gamma^{d_1+d_2}_A(B).$$

for every A-algebra B such that for any homomorphism $B\to C$ of A-algebras the following diagram is commutative

$$\begin{array}{ccc}
\Gamma^{N}_{\Gamma^{d_1+d_2}_A(B)}\big(\Gamma^{d_1}_A(B)\otimes_A \Gamma^{d_2}_A(B)\big) & \xrightarrow{\Lambda^{d_1,d_2}_B} & \Gamma^{d_1+d_2}_A(B) \\
\downarrow & & \downarrow \\
\Gamma^{N}_{\Gamma^{d_1+d_2}_A(C)}\big(\Gamma^{d_1}_A(C)\otimes_A \Gamma^{d_2}_A(C)\big) & \xrightarrow{\Lambda^{d_1,d_2}_C} & \Gamma^{d_1+d_2}_A(C).
\end{array}$$

Proof. If C is an arbitrary A-algebra and B is a flat A-algebra with a surjection $B\twoheadrightarrow C$ then the vertical arrows of the square are surjective and the upper arrow Λ^{d_1,d_2}_B is given by Definition (4.4). We will verify that the composition of the upper and right arrows factors through the left arrow and thus induces a unique homomorphism Λ^{d_1,d_2}_C. As the diagram is commutative for flat A-algebras, it is then easily seen that this definition of Λ^{d_1,d_2}_C is independent on the choice of flat resolution $B\twoheadrightarrow C$ and that the diagram becomes commutative for any homomorphism $B\to C$.

Let I be the kernel of $B\twoheadrightarrow C$. The kernel of the left arrow in the diagram

$$\Gamma^{N}_{\Gamma^{d_1+d_2}_A(B)}\big(\Gamma^{d_1}_A(B)\otimes_A \Gamma^{d_2}_A(B)\big) \twoheadrightarrow \Gamma^{N}_{\Gamma^{d_1+d_2}_A(C)}\big(\Gamma^{d_1}_A(C)\otimes_A \Gamma^{d_2}_A(C)\big)$$

is the $\Gamma^{d_1+d_2}_A(B)$-module generated by the elements

$$\gamma^a\big((\gamma^{b_1}(i)\times f)\otimes(\gamma^{b_2}(j)\times g)\big)\times h$$

with $a\geq 1$, $b_1+b_2\geq 1$, $i,j\in I$, $f\in\Gamma^{d_1-b_1}_A(B)$, $g\in\Gamma^{d_2-b_2}_A(B)$ and $h\in\Gamma^{N-a}_{\Gamma^{d_1+d_2}_A(B)}\big(\Gamma^{d_1}_A(B)\otimes_A \Gamma^{d_2}_A(B)\big)$ by [Ryd08a, 1.2.10]. Furthermore, replacing A with a faithfully flat extension we can assume that f,g and h are of the form $f=\gamma^{d_1-b_1}(x)$, $g=\gamma^{d_2-b_2}(y)$ and $h=\gamma^{N-a}(z)$ where $x,y\in B$

and $z \in \Gamma^{d_1}_A(B) \otimes_A \Gamma^{d_2}_A(B)$ [Fer98, Lem. 2.3.1]. Finally, replacing A with $A[t,u,v]$, it is enough to show that the elements

$$\gamma^N\big(\gamma^{d_1}(i+tx) \otimes \gamma^{d_2}(j+uy) + vz\big)$$
$$\gamma^N\big(\gamma^{d_1}(tx) \otimes \gamma^{d_2}(uy) + vz\big)$$

of $\Gamma^N_{\Gamma^{d_1+d_2}_A(B)}\big(\Gamma^{d_1}_A(B) \otimes_A \Gamma^{d_2}_A(B)\big)$ have the same image in $\Gamma^{d_1+d_2}_A(C)$. This follows by an easy computation. □

(4.6) *Canonical homomorphism* — Next, we consider the canonical homomorphism defined by Iversen in [Ive70, Prop. I.1.5]. This is the homomorphism

$$\omega\,:\,\Gamma^d_A(B) \otimes_A B \to \Gamma^{d-1}_A(B) \otimes_A B$$

given by $\rho_{d-1,1} \otimes \mathrm{id}_B$ followed by the multiplication map. In particular $\omega(\gamma^d(f) \otimes g) = \gamma^{d-1}(f) \otimes fg$. Furthermore, we let

$$u\,:\,\Gamma^d_{\Gamma^d_A(B)}\Big(\Gamma^d_A(B) \otimes_A B\Big) \xrightarrow{\cong} \Gamma^d_A(B) \otimes_A \Gamma^d_A(B) \to \Gamma^d_A(B)$$

be the composition of the canonical base-change isomorphism followed by the multiplication map. This is the homomorphism corresponding to the universal law U given in the beginning of this section.

Proposition (4.7) ([Ive70, Prop. I.1.5])**.** *The homomorphism ω is surjective.*

Proof. It is enough to show that elements of the form $\big(\gamma^{d-1-k}(1) \times x\big) \otimes 1$, where $0 \leq k \leq d-1$ and $x \in \Gamma^k_A(B)$, are in the image of ω. When $k=0$ this is clear. We proceed by induction on k. The element $\big(\gamma^{d-k}(1) \times x\big) \otimes 1 \in \Gamma^d_A(B) \otimes_A B$ is mapped onto an element of the form

$$\big(\gamma^{d-1-k}(1) \times x\big) \otimes 1 + \sum_\alpha \big(\gamma^{d-k}(1) \times y_\alpha\big) \otimes z_\alpha$$

by the formula (4.1.1). By the induction hypothesis it follows that the second term belongs to the image of ω and hence so does the first term. □

The following Proposition generalizes [Ive70, Prop. I.3.1].

Proposition (4.8). *We have that $u = \Lambda^{d-1,1} \circ \Gamma^d_A(\omega)$.*

Proof. Let $u' = \Lambda^{d-1,1} \circ \Gamma^d(\omega)$. As u and u' are $\Gamma^d_A(B)$-algebra homomorphisms, it is enough to show that u and u' coincides on elements of the form $\gamma^{d_1}(1 \otimes b_1) \times \cdots \times \gamma^{d_k}(1 \otimes b_k)$. Replacing A with the polynomial ring $A[t_1,t_2,\dots,t_k]$, it is further enough to show that u and u' coincides on the element $\gamma^d(1 \otimes b') = \gamma^d\big(1 \otimes (t_1b_1 + t_2b_2 + \cdots + t_kb_k)\big)$. This is clear as $\omega(1 \otimes b') = 1 \otimes b'$ and $\Lambda^{d-1,1}(\gamma^d(1 \otimes b')) = \gamma^d(b')$. □

5. The universal family

To abbreviate, we use the notation

$$\Gamma^d_1(X/S) = \Gamma^{d-1}(X/S) \times_S X.$$

as in the introduction. This should be thought of as the space parameterizing zero cycles of degree d with one marked point. The addition morphism

$\Gamma^d_1(X/S) \to \Gamma^d(X/S)$, which we will denote by $\Phi^d_{X/S}$, corresponds to forgetting the marking of the point. We will denote the projection on the marked point $\Gamma^d_1(X/S) \to X$ by π_d. When X/S is affine, we let $\varphi_{X/S}$ be the family of zero cycles of degree d on $\Gamma^d_1(X/S)$ parameterized by $\Gamma^d(X/S)$ given by the shuffle homomorphism $\Lambda^{d-1,1}$ of Proposition (4.5). If a geometric point $\alpha \in \Gamma^d(X/S)$ corresponds to the cycle $x_1 + x_2 + \cdots + x_d$ then $(\varphi_{X/S})_\alpha$ corresponds to the cycle $(x_2 + \cdots + x_{d-1}, x_1) + \cdots + (x_1 + \cdots + x_{d-1}, x_d)$.

(5.1) Let X/S and U/T be separated algebraic spaces. For any commutative diagram

(5.1.1)
$$\begin{array}{ccccc} U & \xrightarrow{f} & X_T & \longrightarrow & X \\ & \searrow & \downarrow & \square & \downarrow \\ & & T & \xrightarrow{g} & S \end{array}$$

there is a natural commutative diagram

(5.1.2)
$$\begin{array}{ccccc} \Gamma^d_1(U/T) & \xrightarrow{\eta} & (f_*)^*\Gamma^d_1(X_T/T) & \longrightarrow & \Gamma^d_1(X_T/T) \\ & \overset{\Phi_{U/T}}{\searrow} & \downarrow & \square & \downarrow{\scriptstyle (\Phi_{X/S})_T} \\ & & \Gamma^d(U/T) & \xrightarrow{f_*} & \Gamma^d(X_T/T) \end{array}$$

Proposition (5.2). *Let X/S be a separated algebraic space. There is a unique family of cycles $\varphi_{X/S}$ of degree d on $\Phi^d_{X/S}$ such that for any commutative diagram* (5.1.1) *with T and U affine, the pull-back of the family $\varphi_{X/S}$ to $\Gamma^d(U/T)$ coincides with the push-forward of $\varphi_{U/T}$ along η.*

Proof. In what follows, all spaces are over T. If $f : U \to X_T$ is any étale morphism then we let $\Gamma^d(U)_{\mathrm{reg}} := \Gamma^d(U/T)|_{\mathrm{reg}(f)}$ be the regular locus [Ryd08a, Cor. 3.3.11]. When $f : U \to X_T$ is étale then the morphism η of diagram (5.1.2) is an isomorphism over $\Gamma^d(U)_{\mathrm{reg}}$ by [Ryd08a, Cor. 3.3.11]. We let $\Gamma^d_1(U)_{\mathrm{reg}} = \Phi_U^{-1}\bigl(\Gamma^d(U)_{\mathrm{reg}}\bigr)$. If $\coprod_\alpha U_\alpha \to X_T$ is an étale cover, then in the diagram

$$\begin{array}{ccccc} \coprod_{\alpha,\beta} \Gamma^d_1(U_\alpha \times_{X_T} U_\beta)_{\mathrm{reg}} & \rightrightarrows & \coprod_\alpha \Gamma^d_1(U_\alpha)_{\mathrm{reg}} & \longrightarrow & \Gamma^d_1(X_T) \\ \downarrow{\scriptstyle \Phi^d_{U_\alpha \times_{X_T} U_\beta}|_{\mathrm{reg}}} & & \downarrow{\scriptstyle \Phi^d_{U_\alpha}|_{\mathrm{reg}}} & & \downarrow{\scriptstyle \Phi^d_{X_T}} \\ \coprod_{\alpha,\beta} \Gamma^d(U_\alpha \times_{X_T} U_\beta)_{\mathrm{reg}} & \rightrightarrows & \coprod_\alpha \Gamma^d(U_\alpha)_{\mathrm{reg}} & \longrightarrow & \Gamma^d(X_T) \end{array}$$

the natural squares are cartesian [Ryd08a, Cor. 3.3.11] and the horizontal sequences are étale equivalence relations [Ryd08a, Cor. 3.3.16]. If we choose a covering such that the U_α's are affine, then we have families $\varphi^d_{U_\alpha \times U_\beta}|_{\mathrm{reg}}$ and $\varphi^d_{U_\alpha}|_{\mathrm{reg}}$ on each component of the two leftmost vertical arrows. By étale descent, we obtain a family $\varphi^d_{X_T}$ on the rightmost arrow. From the compatibility of φ^d with respect to base change and morphisms stated in Proposition (4.5), we can glue the families $\varphi^d_{X_T}$ for every T to a family φ^d_X with the ascribed properties. $\square$

Proposition (5.3). *The morphism* $(\Phi_{X/S}, \pi_d) : \Gamma^d_1(X/S) \to \Gamma^d(X/S) \times_S X$ *is a closed immersion.*

Proof. Follows from Proposition (4.7). □

Proposition (5.4). *Let* X/S *be a separated algebraic space. The family* $\bigl(\Gamma^d_1(X/S), \varphi_{X/S}\bigr)$ *is a representative for the universal family of* $\Gamma^d(X/S)$.

Proof. We have to prove that the composition of the maps

$$\begin{aligned}
\varphi_{X/S} &: \Gamma^d(X/S) \to \Gamma^d\bigl(\Gamma^d_1(X/S)/\Gamma^d(X/S)\bigr) \\
\Gamma^d(\Phi_{X/S}, \pi_d) &: \Gamma^d\bigl(\Gamma^d_1(X/S)\bigr) \hookrightarrow \Gamma^d\bigl(\Gamma^d(X/S) \times_S X\bigr) \\
\pi &: \Gamma^d\bigl(\Gamma^d(X/S) \times_S X\bigr) = \Gamma^d(X/S) \times_S \Gamma^d(X/S) \to \Gamma^d(X/S)
\end{aligned}$$

is the identity. This follows from Proposition (4.8). □

Remark **(5.5).** In general, we do not have that $\Gamma^d_1(X/S) = \mathrm{Image}(\varphi_{X/S})$. It is easily seen however that $\Gamma^d_1(X/S)_{\mathrm{red}} = \mathrm{Supp}(\varphi_{X/S})$.

Proposition (5.6). *The universal family* $\Gamma^d_1(X/S)$ *is étale of rank* d *over* $\Gamma^d(X/S)_{\mathrm{nondeg}}$.

Proof. This is a special case of [Ryd08a, Prop. 4.1.8]. □

Corollary (5.7). *Let* X/S *be a separated algebraic space,* T *an* S*-space and* $\alpha \in \underline{\Gamma}^d_{X/S}(T)$ *a family of cycles. If* α *is non-degenerate at* $t \in T$ *then there is an open neighborhood* $U \ni t$ *such that* $\mathrm{Image}(\alpha|_U) \to U$ *is étale of degree* d*. In particular, the non-degeneracy locus of* α *is open in* T*. Moreover,* $\alpha|_U$ *is given by the canonical family* $\mathcal{N}_{\mathrm{Image}(\alpha|_U)}$ *and the image of* $\alpha|_U$ *commutes with arbitrary base change.*

Proof. Follows immediately from Propositions (3.2) and (5.6). □

Proposition (5.8). *Let* X/S *be a separated family of smooth curves, i.e.,* X/S *is a separated algebraic space, smooth of relative dimension one. Then the universal family* $\Phi^d_{X/S}$ *is locally free of rank* d *and generically étale.*

Proof. The spaces $\Gamma^d_1(X/S)$ and $\Gamma^d(X/S)$ are smooth of relative dimension d over S [Ryd08a, Prop. 4.3.3]. In particular, they are flat over S and we can check the statements about $\Phi^d_{X/S}$ on the fibers. Replacing S with a point s we can thus assume that S is a point. Then $\Gamma^d(X/S)$ and $\Gamma^d_1(X/S)$ are regular and in particular Cohen-Macaulay. As $\Phi^d_{X/S}$ is finite it follows that $\Phi^d_{X/S}$ is flat, cf. [EGA$_{\mathrm{IV}}$, Prop. 15.4.2], and hence locally free. Moreover the connected components of $(X/S)^d$ are irreducible and their generic points are outside the diagonals. Thus $\Phi^d_{X/S}$ is generically étale of rank d, cf. Proposition (5.6). It follows that $\Phi^d_{X/S}$ is locally free of constant rank d. □

6. The Grothendieck-Deligne norm map

In this section we briefly discuss the natural morphism $\mathrm{Hilb}^d(X/S) \to \Gamma^d(X/S)$ which takes a flat subscheme to its norm family. We will call this map the Grothendieck-Deligne norm map as it is introduced in [FGA, No. 221, §6] and [Del73, 6.3.4]. This morphism is closely related to the

Hilbert-Chow morphism [GIT, 5.4] and the Hilbert-Sym morphism [Nee91] as discussed in [Ryd08b].

Definition (6.1). Let $f : X \to S$ be a separated algebraic space and T an S-space. Let $\mathrm{Qcohpf}(X/S)(T)$ be the set of isomorphism classes of quasi-coherent finitely presented $\mathcal{O}_X$-modules which are flat and have proper support over T. We let $\mathrm{Qcohpf}^d(X/S)(T)$ be the subset of $\mathrm{Qcohpf}(X/S)(T)$ consisting of modules $\mathcal{G}$ with support finite over T such that $f_*\mathcal{G}$ is locally free of constant rank d.

The usual pull-back makes $\mathrm{Qcohpf}(X/S)$ and $\mathrm{Qcohpf}^d(X/S)$ into contravariant functors. It can be shown that $\mathrm{Qcohpf}(X/S)$ is the coarse functor to an algebraic stack [LMB00, Thm. 4.6.2.1] but we will not use this.

We have natural transformations

$$\mathrm{Hilb}^d(X/S) \to \mathrm{Qcohpf}^d(X/S)$$
$$\mathrm{Quot}^d(\mathcal{F}/X/S) \to \mathrm{Qcohpf}^d(X/S)$$
$$\mathrm{Qcohpf}^d(X/S) \to \Gamma^d(X/S)$$

where the first two are forgetful morphisms and the last is given by $\mathcal{G} \mapsto \mathcal{N}_{\mathcal{G}}$. Here $\mathcal{N}_{\mathcal{G}}$ is the canonical family determined by $\mathcal{G}$ defined in (3.1). This gives morphisms $\mathrm{Hilb}^d(X/S) \to \Gamma^d(X/S)$ and $\mathrm{Quot}^d(\mathcal{F}/X/S) \to \Gamma^d(X/S)$.

When the canonical family is flat of rank d and generically étale, the morphism $\mathrm{Hilb}^d(X/S) \to \Gamma^d(X/S)$ is an isomorphism [Ive70, Thm. II.3.4]. In particular $\mathrm{Hilb}^d(X/S) \to \Gamma^d(X/S)$ is an isomorphism over $\Gamma^d(X/S)_{\mathrm{nondeg}}$ and an isomorphism if X/S is a family of smooth curves, cf. Propositions (5.6) and (5.8)

7. Composition of families and étale projections

(7.1) *Universal composition of laws* — We have a polynomial law $M \mapsto \Gamma^e_A(\Gamma^d_A(M))$ given by $x \mapsto \gamma^e(\gamma^d(x))$. This law is homogeneous of degree de and thus gives a homomorphism

$$\kappa_{d,e} : \Gamma^{de}_A(M) \to \Gamma^e_A(\Gamma^d_A(M)).$$

Let M, N and P be A-modules. Given polynomial laws $F : M \to N$ and $G : N \to P$ homogeneous of degrees d and e respectively, we form the composite polynomial law $G \circ F : M \to P$. If $f : \Gamma^d_A(M) \to N$, $g : \Gamma^e_A(N) \to P$ and $g * f : \Gamma^{de}_A(M) \to P$ are the corresponding homomorphisms, we have that $g * f = g \circ \Gamma^e(f) \circ \kappa_{d,e}$.

When M, N and P are A-algebras, then $\kappa_{d,e}$ is an algebra homomorphism as the polynomial law defining $\kappa_{d,e}$ is multiplicative. When B is an A-algebra and C a B-algebra, it is also convenient to let $\kappa_{d,e}$ be the natural map

$$\Gamma^{de}_A(C) \to \Gamma^e_A(\Gamma^d_A(C)) \to \Gamma^e_A(\Gamma^d_B(C)).$$

This is the universal composition of a multiplicative law $F : C \to B$ over B which is homogeneous of degree d and a multiplicative law $G : B \to A$ which is homogeneous of degree e.

Definition (7.2). Let X/Y and Y/S be separated algebraic spaces. Let T be an S-space and $\alpha \in \underline{\Gamma}^d_{X/Y}(Y \times_S T)$ and $\beta \in \underline{\Gamma}^e_{Y/S}(T)$ be families of cycles.

Let $Z_\alpha = \mathrm{Image}(\alpha) \hookrightarrow X \times_S T$ and $Z_\beta = \mathrm{Image}(\beta) \hookrightarrow Y \times_S T$. Composing the corresponding laws, we obtain a morphism

$$T \to \Gamma^{de}(Z_\alpha \times_{Y \times_S T} Z_\beta / T) \hookrightarrow \Gamma^{de}(X \times_S T/T)$$

and we let $\beta * \alpha \in \underline{\Gamma}^{de}_{X/S}(T)$ be the corresponding family. By definition $\mathrm{Image}(\beta * \alpha) \hookrightarrow \mathrm{Image}(\alpha) \times_{Y \times_S T} \mathrm{Image}(\beta)$. It is clear that the composition $(\alpha, \beta) \to \beta*\alpha$ is functorial in T and hence we obtain a natural transformation

$$* : \underline{\Gamma}^{e}_{Y/S}(-) \times \underline{\Gamma}^{d}_{X/Y}(Y \times_S -) \to \underline{\Gamma}^{de}_{X/S}(-).$$

of functors from S-schemes to sets. We define $\beta * \alpha$ for inhomogeneous families similarly.

Proposition (7.3). *Let X/Y, Y/S be separated algebraic spaces. Let T be an S-space and let $\alpha \in \underline{\Gamma}^{\star}_{X/Y}(Y \times_S T)$ and $\beta \in \underline{\Gamma}^{\star}_{Y/S}(T)$ be families of cycles.*

(i) *If $f : X \to X'$ is a Y-morphism, then*

$$f_*(\beta * \alpha) = \beta * f_*\alpha.$$

(ii) *Let $g : Y' \to Y$ be an S-morphism and $g' : X' \to X$ be the pull-back of g along X/Y. Let $\beta' \in \underline{\Gamma}^{\star}_{Y'/S}(T)$ be a family of cycles. Then*

$$(g_*\beta') * \alpha = g'_*(\beta' * g^*\alpha)$$

(iii) *If $\alpha' \in \underline{\Gamma}^{\star}_{X/Y}(Y \times_S T)$ and $\beta' \in \underline{\Gamma}^{\star}_{Y/S}(T)$ are families of cycles, then*

$$(\beta + \beta') * \alpha = \beta * \alpha + \beta' * \alpha$$

$$\beta * (\alpha + \alpha') = \beta * \alpha + \beta * \alpha'.$$

Proof. (i) and (ii) are easily verified and (iii) follows from (i) and (ii). □

Remark **(7.4).** Let $S = \mathrm{Spec}(\overline{k})$ where $\overline{k}$ is an algebraically closed field. Let X/Y and Y/S be algebraic spaces with families of cycles α and β of degrees d and e respectively. Then $\beta = y_1 + y_2 + \cdots + y_e$ and $\beta * \alpha = \alpha_{y_1} + \alpha_{y_2} + \cdots + \alpha_{y_e}$.

Proposition (7.5). *Let $f : X \to S$ be a separated morphism, let $g : Y \to S$ be a finite and étale morphism and let $\alpha : S \to \Gamma^\star(X/S)$ be a family of zero cycles. Then $\mathcal{N}_{Y/S} * g^*\alpha = \alpha * \mathcal{N}_{X \times_S Y/X}$.*

Proof. It is enough to show the equality after a faithfully flat base change. We can thus assume that $Y = S^{\amalg n}$ is a trivial cover. Then both sides of the identity are equal to $\alpha_1 + \alpha_2 + \cdots + \alpha_n$ where α_i is the family α on the i^{th} component of $X \times_S Y = X^{\amalg n}$. □

Remark **(7.6).** The following generalization of Proposition (7.5) is probably true. Let $f : X \to S$ and $g : Y \to S$ be separated morphisms and let $\alpha : S \to \Gamma^\star(X/S)$ and $\beta : S \to \Gamma^\star(Y/S)$ be families of zero cycles. Then $\beta * g^*\alpha = \alpha * f^*\beta$.

Proposition (7.7). *Let $Y \to S$ be a finite étale morphism and $X \to Y$ a separated morphism. Then the morphism of presheaves*

$$\underline{\Gamma}^{\star}_{X/Y}(Y \times_S -) \to \underline{\Gamma}^{\star}_{X/S}(-)$$

given by $\alpha \mapsto \mathcal{N}_{Y\times_S -} * \alpha$*, is an isomorphism. In particular, if* Y *and* S *are connected then the degree of any family* $\alpha' \in \underline{\Gamma}^\star_{X/S}(T)$ *is a multiple of the rank of* $Y \to S$*.*

Proof. As the presheaves are sheaves in the étale topology, we can replace S with an étale cover and assume that $Y = S^{\amalg n}$ is a trivial étale cover. We then have a corresponding decomposition $X = \coprod_{i=1}^n X_i$ and any family $\alpha' \in \underline{\Gamma}^\star_{X/S}(T)$ decomposes as a sum $\alpha' = \sum_{i=1}^n \alpha'_i$ where α'_i is supported on $X_i \times_S T$. This gives a family $\alpha = (\alpha'_i) \in \underline{\Gamma}^\star_{X/Y}(Y \times_S T)$ which composed with the canonical family $\mathcal{N}_{Y\times_S T}$ is α'. □

For completeness, we mention the globalization of (7.1).

Definition (7.8) (Universal composition of families)**.** Let X/Y and Y/S be separated algebraic spaces and let d and e be positive integers. Consider the natural projection morphisms

$$\Gamma^e(\Gamma^d(X/Y)/S) \times_S \Gamma^d(X/Y) \times_Y X \\ \to \Gamma^e(\Gamma^d(X/Y)/S) \times_S \Gamma^d(X/Y) \to \Gamma^e(\Gamma^d(X/Y)/S).$$

On the first morphism, we have the family $\mathrm{id}_{\Gamma^e(\Gamma^d(X/Y)/S)} \times_S \Phi^d_{X/Y}$ and on the second we have the family $\Phi^e_{\Gamma^d(X/Y)/S}$. The composition of these families gives a morphism

$$\kappa' : \Gamma^e(\Gamma^d(X/Y)/S) \to \Gamma^{de}(\Gamma^d(X/Y) \times_Y X/S).$$

We let

$$\kappa^{d,e}_{X/Y/S} : \Gamma^e(\Gamma^d(X/Y)/S) \to \Gamma^{de}(X/S)$$

be κ' followed by the push-forward along the projection on the second factor.

Proposition (7.9). *Let* X/Y*,* Y/S *be separated algebraic spaces,* T *an* S*-space and let* $\alpha \in \underline{\Gamma}^d_{X/Y}(Y \times_S T)$ *and* $\beta \in \underline{\Gamma}^e_{Y/S}(T)$ *be families of cycles. Then*

$$\beta * \alpha = \kappa^{d,e} \circ \Gamma^e(\alpha) \circ (\beta, \mathrm{id}_T).$$

Proof. Replacing X and Y with $X \times_S T$ and $Y \times_S T$ we can assume that $T = S$. Let $\widetilde{\beta}$ be the pull-back of the universal family $\Phi^e_{\Gamma^d(X/Y)/S}$ along $\Gamma^e(\alpha)\circ\beta$. Note that $\kappa' \circ \Gamma^e(\alpha) \circ \beta$ corresponds to the family $\widetilde{\beta} * \Phi^d_{X/Y}$. As $\widetilde{\beta}$ is the push-forward of β along the closed immersion $\alpha : Y \to \Gamma^d(X/Y)$, we have that $\widetilde{\beta} * \Phi^d_{X/Y}$ is the push-forward of $\beta * \alpha$ along $\alpha \times_Y \mathrm{id}_X : X \hookrightarrow \Gamma^d(X/Y) \times_Y X$. As $\kappa^{d,e}$ is the push-forward of κ' along the projection $\Gamma^d(X/Y) \times_Y X \to X$, this ends the demonstration. □

8. Families of zero cycles over reduced parameter spaces

The geometric points of $\Gamma^d(X/S)$ correspond to cycles of degree d. To be precise, if k is an algebraically closed field and s is a k-point of S, then the k-points of $\Gamma^d(X/S)$ over s corresponds to the effective zero cycles of degree d on $(X_s)_{\mathrm{red}}$ [Ryd08a, Cor. 3.1.9]. To determine the k-points for an arbitrary field k, we have to characterize the $\overline{k}$-points which descends to k. If k is perfect, these points are the ones corresponding to cycles invariant under the action of the Galois group $\mathrm{Gal}(\overline{k}/k)$. The k-points of $\Gamma^d(X/S)$ are thus effective zero cycles of degree d on $(X_s)_{\mathrm{red}}$ where the degree is counted with multiplicity. The inseparable case is slightly more complicated.

Definition (8.1). Let $k \hookrightarrow K$ be a finite algebraic extension. There is then a canonical factorization into a separable extension $k \hookrightarrow k_s$ and a purely inseparable extension $k_s \hookrightarrow K$. The *separable degree* of K/k is $[k_s : k]$ and the *inseparable degree* is $[K : k_s]$. The *exponent* of K/k is the smallest positive integer n such that $K^n k$ is separable over k, i.e., the smallest positive integer n such that $K^n \subseteq k_s$. We let the *quasi-degree* of K/k be the product of the separable degree and the exponent. We let the *inseparable discrepancy* be the quotient of the inseparable degree with the exponent.

Remark **(8.2).** If k is of characteristic zero, then the inseparable degree, the exponent and the inseparable discrepancy are all one. If k is of characteristic p, then the inseparable degree, the exponent and the inseparable discrepancy are powers of p. Let d_s be the separable degree, d_i the inseparable degree, p^e the exponent, $d = [K : k]$ the degree, d_q the quasi-degree and δ the inseparable discrepancy. Then

$$d = d_s d_i, \quad d_i = p^e \delta, \quad d_q = d_s p^e, \quad d = d_q \delta.$$

The inseparable discrepancy is one if and only if $k_s \hookrightarrow K$ is generated by one element, or equivalently, if and only if $k \hookrightarrow K$ is generated by one element.

Example (8.3). The standard example of a field extension with exponent different from the inseparable degree is the following: Let $k = \mathbb{F}_p(s,t)$ and $K = k^{1/p} = k(s^{1/p}, t^{1/p})$. Then K/k has inseparable degree p^2 and exponent p.

Lemma (8.4). *Let $k \hookrightarrow K$ be a finite algebraic extension of fields of characteristic p. The exponent of K/k is the smallest power p^e such that $k^{p^{-e}} \hookrightarrow k^{p^{-e}}K$ is separable.*

Proof. Standard results on p-bases, cf. [Mat86, Thm. 26.7], show that if $k \hookrightarrow k'$ is a separable algebraic extension then $k^{p^{-e}}k' = k'^{p^{-e}}$. Thus $k^{p^{-e}} \hookrightarrow k^{p^{-e}}K$ is separable if and only if $k_s^{p^{-e}} \hookrightarrow k_s^{p^{-e}}K$ is separable. This is equivalent to $K^{p^e} \subseteq k_s$, i.e., that K/k has exponent at most p^e. □

The following proposition is a reinterpretation of [Kol96, Thm. I.4.5] as will be seen in Proposition (8.13).

Proposition (8.5). *Let $k \hookrightarrow K$ be a finite algebraic extension with quasi-degree d. Then k is equal to the intersection of all purely inseparable extensions k'/k such that $k' \hookrightarrow Kk'$ has degree at most d.*

Proof. Let d_s and p^e be the separable degree and exponent of K/k. Let k_1 be the intersection of all fields k' such that k'/k is purely inseparable and $k' \hookrightarrow Kk'$ has degree at most $d = d_s p^e$. If $k \neq k_1$ we can find an element $x \in k_1 \setminus k$ such that $x^p \in k$. Let k' be a maximal purely inseparable extension of k such that $x \notin k'$. Then $k^{p^{-e}}k' \subseteq k'(x^{p^{-e}})$ by [Kol96, Main Lemma I.4.5.5]. In particular the degree of $k^{p^{-e}}k'/k'$ is at most p^e. Note that by Lemma (8.4) we have that $k^{p^{-e}} \hookrightarrow k^{p^{-e}}K$ is separable and hence has degree d_s. Thus Kk'/k' has degree at most $d_s p^e$. This implies that $x \notin k_1$ which is a contradiction. □

Proposition (8.6). *Let $k \hookrightarrow K$ be a finite field extension. Then $\Gamma^d(K/k)$ has at most one k-point. It has a k-point if and only if the quasi-degree of K/k divides d. This k-point corresponds to the composition of the polynomial laws*

$$F_{\text{insep}} : K \to k_s, \quad b \mapsto b^{d/d_s}$$
$$F_{\text{sep}} : k_s \to k, \quad b \mapsto \mathrm{N}_{k_s/k}(b)$$

where d_s is the separable degree of K/k and $\mathrm{N}_{k_s/k} : k_s \to k$ is the norm, cf. §3. In particular, there is a k-point if $[K : k] \mid d$.

Proof. Let d_s and p^e be the separable degree and the exponent of K/k and k_s its separable closure. By Proposition (7.7) there is a one-to-one correspondence between k-points of $\Gamma^d(K/k)$ and k_s-points of $\Gamma^{d/d_s}(K/k_s)$. Replacing k with k_s and d with d/d_s we can thus assume that K/k is separably closed.

Let $F : K \to k$ be a polynomial law, homogeneous of degree d. Then $K^{p^e} \subseteq k$ and as F is multiplicative we have that $F(b)^{p^e} = F\left(b^{p^e}\right) = \left(b^{p^e}\right)^d$ for any $b \in K$. As p^{th} roots are unique in k it follows that $F(b) = b^d \in k$. As K/k is purely inseparable, it follows that $p^e \mid d$. □

Definition (8.7). Let X/S be a separated algebraic space. Given a family of zero cycles α on X/S parameterized by an S-space T, we define the *multiplicity* of α at a point $x \in X \times_S T$, denoted $\mathrm{mult}_x(\alpha)$, as follows. Let $t \in T$ be the image of x in T. The pull-back of the family to $k(t)$ is then supported at $\mathrm{Image}(\alpha_t) = \mathrm{Supp}(\alpha_t) = \{x_1, x_2, \dots, x_n\}$ and given by the morphism

$$\alpha_t : \mathrm{Spec}\big(k(t)\big) \to \Gamma^d\big(\mathrm{Supp}(\alpha_t)\big) = \coprod_{d_1+d_2+\dots+d_n=d} \mathop{\times}_{i=1}^{n} \Gamma^{d_i}\big(\mathrm{Spec}(k(x_i))\big).$$

As each of the schemes $\Gamma^{d_i}\big(\mathrm{Spec}(k(x_i))\big)$ has at most one $k(t)$-point by Proposition (8.6), the morphism α_t is uniquely determined by the decomposition $d = d_1 + d_2 + \dots + d_n$. The multiplicity at x_i is defined to be $d_i/[k(x_i) : k(t)]$ and zero at points outside $\mathrm{Supp}(\alpha)$. As the support commutes with base change we have that

$$\mathrm{Supp}(\alpha) = \{x \in X \times_S T \; : \; \mathrm{mult}_x(\alpha) > 0\}.$$

Definition (8.8). Let X/S be a separated algebraic space and let T be a S-space. Given a family of zero cycles α on X/S parameterized by T, we let its *fundamental cycle* $[\alpha]$ be the cycle on $X \times_S T$ with coefficients in $\mathbb{Q}$ given by

$$[\alpha] = \sum_{x \in X \times_S (T_{\max})} \mathrm{mult}_x(\alpha) \left[\overline{\{x\}}\right]$$

where $T_{\max}$ is the set of generic points of T.

Proposition (8.9). *Let X/S be a separated algebraic space and T a* reduced *S-space. A family of zero cycles $\alpha \in \underline{\Gamma}^d_{X/S}(T)$ is then uniquely determined by its fundamental cycle $[\alpha]$. Moreover* $\mathrm{Supp}(\alpha) = \mathrm{Supp}\big([\alpha]\big)$.

Proof. As every component of $Z = \mathrm{Supp}(\alpha)$ dominates a component of T [Ryd08a, Thm. 2.4.6], the support of $[\alpha]$ coincides with the support of α. As T is reduced, the morphism $\alpha : T \to \Gamma^d(X/S)$ is determined by its restriction to the generic points of T. If $\xi \in T_{\max}$ then $\alpha_\xi : k(\xi) \to \Gamma^d(\mathrm{Supp}(\alpha_\xi))$ is determined by the multiplicities at the points of $\mathrm{Supp}(\alpha_\xi)$ by Proposition (8.6). □

Definition (8.10). Let k be a field and X/k a separated algebraic space. Let $\mathcal{Z}$ be a zero cycle on X with coefficients in $\mathbb{Q}$. The *degree* of $\mathcal{Z}$ at a point $z \in \mathrm{Supp}(\mathcal{Z})$ is the product of the multiplicity of $\mathcal{Z}$ at z and $[k(z) : k]$. We say that $\mathcal{Z}$ is quasi-integral if for any $z \in \mathrm{Supp}(\mathcal{Z})$ the following two equivalent conditions are satisfied

(i) The product of $\mathrm{mult}_z(\mathcal{Z})$ and the inseparable discrepancy of $k(z)/k$ is an integer.
(ii) The degree of $\mathcal{Z}$ at z is an integer multiple of the quasi-degree of $k(z)/k$.

Note that if k is perfect then $\mathcal{Z}$ is quasi-integral if and only if it has integral coefficients.

Proposition (8.11). *Let k be a field and X/k a separated algebraic space. There is a one-to-one correspondence between k-points of $\Gamma^d(X/k)$ and quasi-integral effective zero cycles on X of degree d. This correspondence takes a family of zero cycles α onto its fundamental cycle $[\alpha]$.*

Proof. Follows from the definitions and Proposition (8.6). □

Definition (8.12). Let k be a field and X/k a separated algebraic space. Let $\mathcal{Z} = \sum_{i=1}^n a_i[Z_i]$ be a zero cycle on X with coefficients in $\mathbb{Q}$. For a field extension k'/k we define the cycle $\mathcal{Z}_{k'}$ on $X_{k'} = X \times_k \mathrm{Spec}(k')$ as

$$\mathcal{Z}_{k'} = \sum_{i=1}^n a_i[Z_i \times_k \mathrm{Spec}(k')]$$

where $[Z_i \times_k \mathrm{Spec}(k')]$ is the fundamental cycle of $Z_i \times_k \mathrm{Spec}(k')$, i.e., the sum of the irreducible components of $Z_i \times_k \mathrm{Spec}(k')$ weighted by the lengths of the local rings at their generic points.

If $\alpha \in \Gamma^d(X/k)$ and k'/k is a field extension, then $[\alpha]_{k'} = [\alpha_{k'}]$.

Proposition (8.13) ([Kol96, Thm. I.4.5])**.** *Let k be a field and X/k a separated algebraic space. Let $\mathcal{Z}$ be a zero cycle on X with coefficients in $\mathbb{Q}$. Then $\mathcal{Z}$ is quasi-integral if and only if k is the intersection of all purely inseparable field extensions $k' \supseteq k$ such that $\mathcal{Z}_{k'}$ has integral coefficients.*

Proof. Follows immediately from Proposition (8.5). □

Remark **(8.14).** It is reasonable that an effective zero cycle on X with integral coefficients should give a family of zero cycles on X/k. The above proposition explains why fractional coefficients are also sometimes allowed. Indeed, let $\mathcal{Z}$ be an effective zero cycle on $X_{\overline{k}}$ with integral coefficients and let α be the corresponding point in $\Gamma^d(X/k)$. If k'/k is a field extensions such that $\mathcal{Z}$ decends to $X \times_k \mathrm{Spec}(k')$ with integral coefficients, then α is defined over k'. Thus the residue field of α has at least to be small enough to be contained in all such field extensions k'. Proposition (8.13) states that the residue field is not smaller than this.

9. Families of zero cycles as multivalued morphisms

In this section, we give an alternative description of families of zero cycles on AF-schemes as "multi-morphisms".

Definition (9.1). A *multivalued map* $f : X \to Y$ is a map which to every $x \in X$ assigns a finite subset $f(x) \subseteq Y$. The *inverse image* of $W \subseteq Y$ with respect to f is

$$f^{-1}(W) = \{x \in X \ : \ f(x) \subseteq W\}.$$

A multivalued map $f : X \to Y$ of topological spaces is *continuous* if $f^{-1}(U)$ is open for every open subset $U \subseteq Y$. A multivalued map $f : X \to Y$ is of *degree at most d* if $|f(x)| \leq d$ for every $x \in X$.

Note that it is allowed for $f(x)$ to be the empty set.

Definition (9.2). Let X be a topological space. A *d-cover* of X is an open cover $\{U_\alpha\}$ of X such that any set of at most d points of X is contained in one of the U_α's. A *d-sheaf* on X is a presheaf $\mathcal{F}$ on X such that

$$\mathcal{F}(U) \longrightarrow \prod_\alpha \mathcal{F}(U_\alpha) \rightrightarrows \prod_{\alpha,\beta} \mathcal{F}(U_\alpha \cap U_\beta)$$

is exact for any open subset $U \subseteq X$ and every d-cover $\{U_\alpha\}$ of U. In other words, a d-sheaf is a sheaf in the Grothendieck topology on X where the covers are d-covers. A 1-sheaf is an ordinary sheaf.

Definition (9.3). Let $f : X \to Y$ be a continuous multivalued map of degree at most d. If $\mathcal{F}$ is a presheaf of sets on X we let $f_*\mathcal{F}$ be the presheaf $U \mapsto \mathcal{F}(f^{-1}(U))$ for every open subset $U \subseteq Y$. If $\mathcal{F}$ is a k-sheaf then $f_*\mathcal{F}$ is a dk-sheaf. If $\mathcal{F}$ is a dk-sheaf of sets on Y we let $f^*\mathcal{F}$ be the associated k-sheaf to the presheaf $U \mapsto \varinjlim_{V \supseteq f(U)} \mathcal{F}(V)$, where $U \subseteq X$ is open and the limit is over all open subsets $V \subseteq Y$ containing $f(U)$. If $\mathcal{F}$ is a presheaf on X and $Z \subseteq X$ is a finite subset, we denote by

$$\mathcal{F}_Z = \varinjlim_{V \supseteq Z} \mathcal{F}(V)$$

the *stalk* at Z.

It is not difficult to see, as in the single-valued case, that if $f : X \to Y$ is a continuous multivalued map of degree at most d, then f^* and f_* are adjoint functors between the categories of k-sheaves on X and kd-sheaves on Y and $(f^*\mathcal{F})_x = \mathcal{F}_{f(x)}$.

Definition (9.4). Let X and Y be ringed spaces. A *multi-morphism* from X to Y is a pair (f, θ) consisting of a multivalued continuous map $f : X \to Y$ and a multiplicative law (of presheaves) $\theta : \mathcal{O}_Y \to f_*\mathcal{O}_X$ over $\mathbb{Z}$. We say that (f, θ) is of degree d if θ is homogeneous of degree d.

Remark **(9.5).** An ordinary morphism of ringed spaces is a multi-morphism of degree 1. Given multi-morphisms $f : X \to Y$ and $g : Y \to Z$ we can form the composition $g \circ f : X \to Z$. If f and g has degrees d and e respectively, then $g \circ f$ has degree de.

Proposition (9.6). *Let $(f, \theta) : X \to Y$ be a multi-morphism of ringed spaces. There is a canonical partition $X = \coprod_{d \geq 0} X_d$ of open and closed subsets $X_d \subseteq X$ such that $f|_{X_d}$ is a multi-morphism of degree d.*

Proof. This follows easily from Proposition (2.2). □

Definition (9.7). Let A be a semi-local ring and B be a local ring. A multiplicative $\mathbb{Z}$-law $A \to B$ is called *semi-local* if the kernel of the composite law $A \to B \to B/\mathfrak{m}_B$ is the Jacobson radical of A.

Note that if Y is an AF-scheme and $Z \subseteq Y$ is finite, then the stalk $\mathcal{O}_{Y,Z}$ is semi-local.

Definition (9.8). Let X and Y be schemes. A *multi-morphism* from X to Y is a multi-morphism of ringed spaces (f, θ) such that $\mathcal{O}_{Y,f(x)}$ is semi-local and the law $\theta_x^\sharp : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is semi-local for every $x \in X$.

Remark **(9.9).** If $f : X \to Y$ and $g : Y \to Z$ are multi-morphisms of schemes, then $g \circ f : X \to Z$ is a multi-morphism of schemes if $\mathcal{O}_{Z,g(f(x))}$ is semi-local for every $x \in X$.

Proposition (9.10). *Let $(f, \theta) : X \to Y$ be a multi-morphism of schemes. If (f, θ) has degree d, then the multivalued map f is of degree at most d. In particular, there is a one-to-one correspondence between multi-morphisms of degree one and ordinary morphisms of schemes.*

Proof. If θ is of degree d then so is $\theta_x^\sharp$. The kernel of $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}/\mathfrak{m}_x$ is by assumption the Jacobson radical $\mathfrak{r}$ of $\mathcal{O}_{Y,f(x)}$. Let $B = \mathcal{O}_{Y,f(x)}/\mathfrak{r}$. We thus have a non-degenerate multiplicative law $B \to k(x)$ of degree d. This law factors through a homomorphism $B \to B \otimes_{\mathbb{Z}} k(x) \twoheadrightarrow B'$ where B' is a product of at most d fields [Ryd08a, Thm. 2.4.6]. As $B \to k(x)$ is non-degenerate, we have by definition that $B \to B'$ is injective and thus B is a product of at most d fields. □

Definition (9.11). Let $f = (f, \theta) : X \to Y$ be a multi-morphism of schemes and let n be a positive integer. We denote by $n \cdot f$ the multi-morphism (f, θ^n) from X to Y where θ^n is the homomorphism θ followed by taking the n^{th} power. If f has degree d then $n \cdot f$ has degree nd. More generally, if $f_1, f_2 : X \to Y$ are multi-morphisms, we can define their sum $f_1 + f_2 : X \to Y$ as the multi-morphism $(f_1 \cup f_2, \theta_1 \theta_2)$. If $\mathcal{O}_{Y, f_1(x) \cup f_2(x)}$ is semi-local, this is a multi-morphism of schemes. If f_1 and f_2 have degrees d_1 and d_2 respectively, then $f_1 + f_2$ has degree $d_1 + d_2$.

Definition (9.12). Let X and Y be S-schemes with structure morphisms $\varphi_X : X \to S$ and $\varphi_Y : Y \to S$. We say that a multi-morphism $(f, \theta) : X \to Y$ is an S-multi-morphism if $\varphi_Y \circ f = \varphi_X$ as multivalued maps and $\theta : \mathcal{O}_Y \to f_* \mathcal{O}_X$ is a $\varphi_Y^* \mathcal{O}_S$-law. Here the $\varphi_Y^* \mathcal{O}_S$-algebra structure on $f_* \mathcal{O}_X$ is given by the homomorphism $\mathcal{O}_S \to (\varphi_X)_* \mathcal{O}_X = (\varphi_Y)_* f_* \mathcal{O}_X$ and adjointness.

Proposition (9.13). *Let $\varphi_X : X \to S$ and $\varphi_Y : Y \to S$ be S-schemes. If $f : X \to Y$ is an S-multi-morphism of degree d, then $\varphi_Y \circ f = d \cdot \varphi_X$.*

Proof. The law defining $\varphi_Y \circ f$ is $\psi : \mathcal{O}_S \to (\varphi_Y)_*\mathcal{O}_Y \to (\varphi_Y)_* f_*\mathcal{O}_X$. As $(\varphi_Y)_*\mathcal{O}_Y \to (\varphi_Y)_* f_*\mathcal{O}_X$ is an $\mathcal{O}_S$-law, it follows that ψ is the d^{th} power of $\mathcal{O}_S \to (\varphi_X)_*\mathcal{O}_X$. □

It is *not* true, unless X is reduced, that if $f : X \to Y$ is a multi-morphism of S-schemes such that $\varphi_Y \circ f = d \cdot \varphi_X$, then f is a S-multi-morphism. This is demonstrated by the following example.

Example (9.14). Let $A = \mathbb{Z}[x]$, $B = A[y]$, $C = A[\epsilon]/\epsilon^2$. Then it can be shown that

$$\Gamma^2_{\mathbb{Z}}(B) = \frac{\mathbb{Z}[x_p, x_s, y_p, y_s, x \times y]}{(x \times y)^2 - x_s y_s (x \times y) + x_p y_s^2 + x_s^2 y_p - 4x_p y_p}$$

where $x_p = \gamma^2(x)$, $x_s = x \times 1$, $y_p = \gamma^2(y)$ and $y_s = y \times 1$. Let $F : B \to C$ be a multiplicative $\mathbb{Z}$-law of degree 2 and let $f : \Gamma^2_{\mathbb{Z}}(B) \to C$ be the corresponding homomorphism. That the composite law $A \to B \to C$ is $a \mapsto a^2 \cdot 1_C$ is equivalent to $f(\gamma^2(x)) = x^2$ and $f(x_s) = 2x$. This implies that

$$\big(f(x \times y) - x f(y_s)\big)^2 = 0.$$

In particular, $f(y_p) = f(y_s) = 0$ and $f(x \times y) = \epsilon$ defines a homomorphism such that $A \to B \to C$ is taking squares. It is clear that F is not an A-law as this would imply that $f(x \times y) = x f(y_s) = 0$.

Theorem (9.15). *Let X/S be any scheme and Y/S be an* AF*-scheme. There is a one-to-one correspondence between S-multi-morphisms $f : X \to Y$ and families of zero cycles, i.e., morphisms $\alpha : X \to \Gamma^\star(Y/S)$. In this correspondence a family of cycles α corresponds to the multi-morphism (f, θ) such that*

(i) *For every $x \in X$, the image $f(x)$ is the projection of the support* $\operatorname{Supp}(\alpha \times_X \operatorname{Spec}(k(x))) \hookrightarrow Y \times_S \operatorname{Spec}(k(x))$ *onto Y.*
(ii) *For any affine open subsets $V \subseteq S$ and $U \subseteq Y \times_S V$, the law*
$$\theta(U) : \mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}(U))$$
corresponds to the morphism
$$\alpha|_{\Gamma^\star(U/V)} : \alpha^{-1}(\Gamma^\star(U/V)) \to \Gamma^\star(U/V).$$

Proof. To begin with, note that for any open $U \subseteq Y$, we have that $f^{-1}(U) = \alpha^{-1}(\Gamma^{\geq 1}(U/S))$. In particular, f is continuous. It is further clear that θ is a morphism of presheaves and that $\theta_x^\sharp$ is the law corresponding to $\operatorname{Spec}(\mathcal{O}_{X,x}) \to \Gamma^d(\operatorname{Spec}(\mathcal{O}_{Y,f(x)}))$ where d is the degree of α at x. This law is semi-local by the definition of f.

We will now construct an inverse to the mapping $\alpha \to (f, \theta)$. For this, we can assume that S is affine and that θ is homogeneous of degree d. As Y is an AF-scheme, there is an open affine cover $\{U_\beta\}$ of Y such that any d points of Y lie in some U_β. This induces an open affine cover $\{\Gamma^d(U_\beta/S)\}$ of $\Gamma^d(Y/S)$ [Ryd08a, Prop. 3.1.10]. The laws $\theta(U_\beta)$ correspond to morphisms $\alpha_\beta : f^{-1}(U_\beta) \to \Gamma^d(U_\beta/S)$. The semi-locality of θ ensures that if $x \in f^{-1}(U_\beta)$ then the projection of $\operatorname{Supp}(\alpha_\beta)_x$ onto U_β is $f(x)$. In particular,

$\alpha_\beta^{-1}\bigl(\Gamma^d(U_{\beta'}/S)\bigr) = f^{-1}(U_\beta \cap U_{\beta'})$. Thus the α_β's glue to a morphism $\alpha : X \to \Gamma^d(Y/S)$ which corresponds to (f, θ). □

10. Sheaves of divided powers

Let X/S be a separated algebraic space and $\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-module. In this section we construct a canonical quasi-coherent sheaf $\Gamma^d(\mathcal{F})$ on $\Gamma^d(X/S)$. This is a globalization of the construction of the $\Gamma^d_A(B)$-module $\Gamma^d_A(M)$ for an A-algebra B and a B-module M. The sheaf $\Gamma^d(\mathcal{F})$ has been constructed by Deligne when X/S is flat [Del73, 5.5.29].

Proposition (10.1). *Let X/S be a separated algebraic space and $\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-module. There is then a canonical quasi-coherent sheaf $\Gamma^d(\mathcal{F})$ on $\Gamma^d(X/S)$. If $f : X' \to X$ is étale, then there is a canonical isomorphism $\Gamma^d(f^{-1}\mathcal{F})|_{\mathrm{reg}(f)} \to (f_*|_{\mathrm{reg}})^{-1}\,\Gamma^d(\mathcal{F})$. If X/S is affine, then $\Gamma^d(\mathcal{F})$ is canonically isomorphic to $\Gamma^d_{\mathcal{O}_S}(\mathcal{F})$.*

Proof. We will construct $\Gamma^d(\mathcal{F})$ through étale descent via the étale equivalence relation

$$\coprod_{\alpha,\beta} \Gamma^d(U_\alpha \times_X U_\beta/S)|_{\mathrm{reg}} \rightrightarrows \coprod_{\alpha} \Gamma^d(U_\alpha/S)|_{\mathrm{reg}} \longrightarrow \Gamma^d(X/S)$$

for an étale covering $\{U_\alpha \to X\}$ [Ryd08a, 3.3.16.1]. If the U_α's are affine then so are the $U_\alpha \times_X U_\beta$'s. The proposition thus follows after we have showed that

$$\Gamma^d_{\mathcal{O}_S}(f^{-1}\mathcal{F})|_{\mathrm{reg}(f)} \to (f_*|_{\mathrm{reg}})^{-1}\Gamma^d_{\mathcal{O}_S}(\mathcal{F}) \tag{10.1.1}$$

is an isomorphism for any étale morphism $f : X' \to X$ of *affine* schemes. Let $Y = \mathrm{V}(\mathcal{F}) = \mathrm{Spec}(\mathrm{S}(\mathcal{F}))$. Then

$$\Gamma^d(Y \times_X X'/S)|_{\mathrm{reg}(f)} \to (f_*|_{\mathrm{reg}})^{-1}\Gamma^d(Y/S) \tag{10.1.2}$$

is an isomorphism [Ryd08a, Cor. 3.3.11]. As $\mathcal{F}$ is a direct summand of $\mathcal{O}_Y$, it follows from (10.1.2) that (10.1.1) is an isomorphism. □

11. Weil restriction and the norm functor

In this section, we globalize and generalize the results of Ferrand on the norm functor [Fer98]. Let $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ be a finite faithfully flat and finitely presented morphism of constant rank d. In this situation Ferrand constructs a norm functor $N_{B/A}$ from B-modules to A-modules which is uniquely determined by the following properties:

(i) $N_{B/A}(B) = A$ and the image of the multiplication by b in B is the multiplication by $\mathrm{N}_{B/A}(b)$ in A, cf. §3.
(ii) The norm functor commutes with base change, i.e., for any A-algebra A', denoting $B' = B \otimes_A A'$, we have that the functors

$$M \mapsto N_{B/A}(M) \otimes_A A' \quad \text{and} \quad M \mapsto N_{B'/A'}(M \otimes_B B')$$

are isomorphic.

The functoriality gives a polynomial law $\nu : M \to N_{B/A}(M)$, homogeneous of degree d, which is compatible with the polynomial law $\mathrm{N}_{B/A}$. If C is a B-algebra then $N_{B/A}(C)$ is an A-algebra. Ferrand constructs $N_{B/A}(M)$ as the tensor product $\Gamma^d_A(M) \otimes_{\Gamma^d_A(B)} A$ where the $\Gamma^d_A(B)$-algebra structure of A is given by the determinant law $\mathrm{N}_{B/A} : B \to A$.

Given algebraic spaces X/S and Y/S together with a family of cycles $\alpha : Y \to \Gamma^\star(X/S)$ we will construct a norm functor $N_\alpha : \mathcal{C}_X \to \mathcal{C}_Y$. Here $\mathcal{C}$ is one of the following fibered categories over the category of algebraic spaces:

- The category of quasi-coherent modules **QCoh**.
- The category of affine schemes **Aff**.
- The category of separated algebraic spaces **AlgSp**.

In Ferrand's setting, $S = Y$ is affine, X/S is finite flat of constant rank d and $\alpha = \mathcal{N}_{X/S}$ is the canonical family given by the determinant, cf. Definition (3.1). We construct the generalized norm functor in the obvious way:

Definition (11.1). With notation as above, we let $N_\alpha(W) = \alpha^*\Gamma^\star(W/S)$ where $W \in \mathcal{C}_X$. If W is an algebraic space, we let $\nu_\alpha(W)$ be the induced family of cycles $\nu_\alpha(W) : N_\alpha(W) \to \Gamma^\star(W/S)$ as in the diagram below:

$$\begin{array}{ccc} \Gamma^\star(W/S) & \xleftarrow{\nu_\alpha(W)} & N_\alpha(W) \\ \downarrow & \square & \downarrow \\ \Gamma^\star(X/S) & \xleftarrow{\quad\alpha\quad} & Y. \end{array}$$

When W is a quasi-coherent $\mathcal{O}_X$-module, we let $\nu_\alpha(W)$ be the induced homomorphism $\Gamma^\star(W) \to \alpha_* N_\alpha(W)$ on $\Gamma^\star(X/S)$.

Remark **(11.2).** When W/X is étale (or unramified) it is possible to define a "regular norm functor" using $N_\alpha(W)_{\mathrm{reg}} = \alpha^*\bigl(\Gamma^\star(W/S)_{\mathrm{reg}}\bigr)$.

Remark **(11.3).** If Z/S is a third space and $\beta : Z \to \Gamma^e(Y/S)$ is a family of cycles, there is a functorial morphism $N_\beta(N_\alpha(W)) \to N_{\alpha\circ\beta}(W)$ but this is not always an isomorphism, cf. [Fer98, Ex. 4.4].

When W/X is a space, it is useful to think of $N_\alpha(W)$ as the pull-back of W along the multi-morphism α as in the following diagram:

$$\begin{array}{ccc} W & \overset{\nu_\alpha(W)}{\leftarrow - -} & N_\alpha(W) \\ \downarrow & & \downarrow \\ X & \overset{\alpha}{\leftarrow - - -} & Y. \end{array}$$

Proposition (11.4). *With notation as above, let W/X be a space and Y' be a Y-scheme. The Y'-points of $N_\alpha(W)$ corresponds to the set of liftings of the family of cycles $\alpha \times_Y Y'$ to a family of cycles $\beta : Y' \to \Gamma^\star(W/S)$. In other words, it is the set of liftings of the multi-morphism $\alpha \times_Y Y'$ to a* *multi-morphism β in the diagram*

$$\begin{array}{ccccc} W & \overset{\nu_\alpha(W)}{\dashleftarrow} & N_\alpha(W) & \longleftarrow & N_\alpha(W)\times_Y Y' \\ \downarrow & & \downarrow & \beta & \downarrow \\ X & \overset{\alpha}{\dashleftarrow} & Y & \longleftarrow & Y'. \end{array}$$

If α is non-degenerate, the lifting β is non-degenerate and the Y'-points of $N_\alpha(W)$ correspond to sections of $W \to X$ over $\mathrm{Image}(\alpha)\times_Y Y' \hookrightarrow X\times_S Y'$*. If W/X is unramified, the Y'-points of $N_\alpha(W)_{\mathrm{reg}}$ correspond to sections of $W \to X$ over* $\mathrm{Image}(\alpha\times_Y Y')$*.*

Proof. The correspondence follows from the construction of $N_\alpha(W)$. The last two assertions are immediate consequences of the definitions of non-degenerate families and regular families [Ryd08a, Defs. 4.1.6 and 3.3.3] taking into account that $\mathrm{Image}(\alpha\times_Y Y') = \mathrm{Image}(\alpha)\times_Y Y'$ when α is non-degenerate, cf. Corollary (5.7). □

Definition (11.5). Let $X \to Y$ and $W \to X$ be morphism of algebraic spaces. The *Weil restriction* $\mathbf{R}_{X/Y}(W)$ is the functor from Y-schemes to sets that takes an Y-scheme Y' to the set of sections of $W\times_Y Y' \to X\times_Y Y'$.

Corollary (11.6) ([Fer98, Prop. 6.2.2])**.** *Let $X \to Y$ be a morphism and $\alpha : Y \to \Gamma^\star(X/Y)$ a family of cycles. Let W be an algebraic space separated over X. There is then a canonical morphism $\mathbf{R}_{X/Y}(W) \to N_\alpha(W)$ which is functorial in W. Assume that $X \to Y$ is finite and étale and that $\alpha = \mathcal{N}_{X/Y}$ is the canonical family given by the determinant. Then the above functor is an isomorphism.*

Proof. Follows immediately from Proposition (11.4) as α is non-degenerate and hence $\mathrm{Image}(\alpha\times_Y Y') = \mathrm{Image}(\alpha)\times_Y Y' = X\times_Y Y'$. □

Corollary (11.7). *Let $f : X \to Y$ be a finitely presented morphism such that there exists a family of zero cycles $\alpha : Y \to \Gamma^\star(X/Y)$ with* $\mathrm{Supp}(\alpha) = X_{\mathrm{red}}$*, e.g., f finite and flat, or Y normal and f finite and open. If W is an étale and separated scheme over X, then $N_\alpha(W)_{\mathrm{reg}}$ coincides with the Weil restriction $\mathbf{R}_{X/Y}(W)$. In particular, the canonical morphism $\mathbf{R}_{X/Y}(W) \to N_\alpha(W)$ is an open immersion.*

Proof. Note that $\mathrm{Image}(\alpha\times_Y Y')$ has the same support as $X\times_Y Y'$. As W/X is étale, any section of W/X over $\mathrm{Supp}(\alpha\times_Y Y')$ thus lifts to a unique section over $X\times_Y Y'$. □

Example (11.8). The following counter-example, due to Ferrand [Fer98, 6.4], shows that even if W/X is finite and étale and X/Y is finite and flat, but not étale, it may happen that $N_\alpha(W)_{\mathrm{reg}} \subseteq N_\alpha(W)$ is not an isomorphism and that $N_\alpha(W) \to Y$ is not étale.

Let $X = \mathrm{Spec}(L) \to Y = \mathrm{Spec}(K)$ correspond to an inseparable field extension $K \subseteq L$ of degree d. Let $W = X^{\amalg d}$. Then there is a closed point in $N_\alpha(W)$ with residue field L. This point corresponds to the family $s_1 + s_2 + \cdots + s_l$ where $s_i : \mathrm{Spec}(L) \to W$ is the inclusion of the i^{th} copy. Thus $N_\alpha(W) \to Y$ is not étale and as $N_\alpha(W)_{\mathrm{reg}} \to Y$ is étale the subset $N_\alpha(W)_{\mathrm{reg}} \subseteq N_\alpha(W)$ is proper.

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