ADDENDUM: ÉTALE DÉVISSAGE, DESCENT AND PUSHOUTS OF STACKS

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Abstract. Using Nisnevich coverings and a Hilbert stack of stacky points, we prove étale dévissage results for non-representable étale and quasi-finite flat coverings. We give applications to absolute noetherian approximation of algebraic stacks and compact generation of derived categories.

1. Introduction

In [Ryd11a, Thm. D & 6.1], dévissage results were proved for representable quasi-finite flat and étale morphisms. We will show how these results may be extended to the non-representable situation using Nisnevich coverings and a Hilbert stack of stacky points.

We apply these results to weaken the separation hypotheses from the approximation results for algebraic stacks that appeared in [Ryd15] and the compact generation result for derived categories of quasi-coherent sheaves on Deligne–Mumford stacks that appeared in [HR17, Thm. A].

The results of this article have already been used in [HK17]. We also expect further applications arising from the work of [AHR15, AHR14] on the local structure of stacks near points with linearly reductive stabilizers, where non-representable étale coverings naturally arise (see Remark 7.6).

Before stating our main result, we require some notation. Fix an algebraic stack S. If P_1, …, P_r is a list of properties of morphisms of algebraic stacks over S, let $\mathbf{Stack}_{P_1,\dots,P_r/S}$ denote the full 2-subcategory of the 2-category of algebraic stacks over S whose objects are those $(x\colon X \to S)$ such that x has properties P_1, …, P_r. The following abbreviations will be used: ét (étale), qff (quasi-finite flat), sep (separated), fp (finitely presented), rep (representable), and sep_Δ (separated diagonal). Throughout, we let $\mathbf{E} \subseteq \mathbf{Stack}_{/S}$ be one of the following 2-subcategories:

$$\begin{array}{ccccc} \mathbf{Stack}_{\mathrm{repr,sep,fp,\acute{e}t}/S} & \subseteq & \mathbf{Stack}_{\mathrm{sep,fp,\acute{e}t}/S} & \subseteq & \mathbf{Stack}_{\mathrm{sep}_\Delta,\mathrm{fp,\acute{e}t}/S} \\ \rotatebox{90}{$\supseteq$} & & \rotatebox{90}{$\supseteq$} & & \rotatebox{90}{$\supseteq$} \\ \mathbf{Stack}_{\mathrm{repr,sep,fp,qff}/S} & \subseteq & \mathbf{Stack}_{\mathrm{sep,fp,qff}/S} & \subseteq & \mathbf{Stack}_{\mathrm{sep}_\Delta,\mathrm{fp,qff}/S}. \end{array}$$

Our improvement of [Ryd11a, Thm. D & 6.1] is the following theorem.

Theorem D′ (Étale or quasi-finite flat dévissage)**.** *Let S be a quasi-compact and quasi-separated algebraic stack and let $\mathbf{E}$ be as above. Let $(T' \xrightarrow{t} T) \in \mathbf{E}$ be surjective (resp. surjective and representable) and let $\mathbf{D} \subseteq \mathbf{E}$ be a full 2-subcategory satisfying the following three conditions:*

(D1) *if $(X' \to X) \in \mathbf{E}$ is étale and $X \in \mathbf{D}$, then $X' \in \mathbf{D}$;*

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(D2) *if* $(X' \to X) \in \mathbf{E}$ *is proper (resp. finite) and surjective and* $X' \in \mathbf{D}$*, then* $X \in \mathbf{D}$*; and*

(D3) *if* $(U \xrightarrow{i} X)$*,* $(X' \xrightarrow{f} X) \in \mathbf{E}$*, where* i *is an open immersion and* f *is étale and an isomorphism over* $X \setminus U$*, then* $X \in \mathbf{D}$ *whenever* U*,* $X' \in \mathbf{D}$*.*

If $T' \in \mathbf{D}$*, then* $T \in \mathbf{D}$*.*

Proof. Combine Theorem 6.1 with Lemma 3.4. □

Note that if $(X' \to X) \in \mathbf{E}$ is étale, then there is a canonical factorization $X' \to X'' \to X$ in $\mathbf{E}$ where the first morphism is an étale gerbe and the second morphism is étale and representable. If in addition $(X' \to X)$ is proper, then $X' \to X''$ is a proper étale gerbe and $X'' \to X$ is finite étale.

Note that if $T' \to T$ is representable, then it has separated diagonal. In particular, the advantage of Theorem D′ over [Ryd11a, Thm. D] is the removal of the assumption of representability from $T' \to T$.

The "Induction principle" [Stacks, Tag 08GL] for algebraic spaces is closely related to the dévissage results of Theorem D′. When working with derived categories or K-theory, where locality results are often quite subtle, it is often advantageous to have the strongest possible criteria at your disposal (e.g., [Hal16]). For stacks with quasi-finite diagonal, we also obtain the following Induction principle.

Theorem E (Induction principle for stacks with quasi-finite diagonal)**.** *Let* S *be a quasi-compact and quasi-separated algebraic stack. Choose* $\mathbf{E} \subseteq \mathbf{Stack}_{/S}$ *as follows:*

- *if* S *has quasi-finite diagonal, take* $\mathbf{E} = \mathbf{Stack}_{\mathrm{sep}_\Delta,\mathrm{fp},\mathrm{qff}/S}$*;*
- *if* S *has quasi-finite and separated diagonal, take* $\mathbf{E} = \mathbf{Stack}_{\mathrm{repr},\mathrm{sep},\mathrm{fp},\mathrm{qff}/S}$*;*
- *if* S *is Deligne–Mumford, take* $\mathbf{E} = \mathbf{Stack}_{\mathrm{sep}_\Delta,\mathrm{fp},\text{ét}/S}$*; and*
- *if* S *is Deligne–Mumford with separated diagonal, take* $\mathbf{E} = \mathbf{Stack}_{\mathrm{repr},\mathrm{sep},\mathrm{fp},\text{ét}/S}$*.*

Let $\mathbf{D} \subseteq \mathbf{E}$ *be a full* 2*-subcategory satisfying the following properties:*

(I1) *if* $(X' \to X) \in \mathbf{E}$ *is an open immersion and* $X \in \mathbf{D}$*, then* $X' \in \mathbf{D}$*;*

(I2) *if* $(X' \to X) \in \mathbf{E}$ *is finite and surjective, where* X' *is an affine scheme, then* $X \in \mathbf{D}$*; and*

(I3) *if* $(U \xrightarrow{i} X)$*,* $(X' \xrightarrow{f} X) \in \mathbf{E}$*, where* i *is an open immersion and* f *is étale and an isomorphism over* $X \setminus U$*, then* $X \in \mathbf{D}$ *whenever* U*,* $X' \in \mathbf{D}$*.*

Then $\mathbf{D} = \mathbf{E}$*. In particular,* $S \in \mathbf{D}$*.*

Proof. Combine Lemma 3.4 with Theorem 4.1. □

We wish to point out that Theorem E relies on the existence of coarse spaces for stacks with finite inertia (i.e., the Keel–Mori Theorem [KM97, Ryd13]). Theorem E, in the case of a separated diagonal, was proved in [Hal16, App. B].

Remark 1.1. Extending Theorem D′ to covers with non-separated diagonals is possible. The most natural and useful formulation, however, requires 2-stacks and the corresponding notion of 2-Nisnevich coverings. This is analogous to the situation of representable but non-separated coverings, where non-representable Nisnevich coverings naturally appear. See Remark 5.4 for more details.

Conventions. We make no a priori separation assumptions on our algebraic stacks, just as in [Stacks].

2. Residual gerbes as intersections

Let X be a quasi-separated algebraic stack (e.g., X noetherian). By [Ryd11a, Thm. B.2], every point of X is algebraic. That is, if $x \in |X|$, then there is a quasi-affine monomorphism $\mathcal{G}_x \to X$ with image x such that $\mathcal{G}_x$ is an fppf gerbe, the *residual gerbe*. Using the recent approximation result [Ryd16], which depends on the original étale dévissage [Ryd11a], we obtain

Lemma 2.1. *Let X be a quasi-separated algebraic stack and let $x \in |X|$ be a point. The residual gerbe $\mathcal{G}_x$ is the limit of an inverse system of immersions $j_\lambda \colon U_\lambda \hookrightarrow X$ of finite presentation with affine bonding maps.*

Proof. There is a locally closed integral substack $Z \hookrightarrow X$ such that Z is a gerbe over an affine scheme $\underline{Z}$ and x is the generic point of Z [Ryd11a, Thm. B.2]. Let $U \subseteq X$ be a quasi-compact open neighborhood of Z such that $Z \hookrightarrow U$ is a closed immersion. Consider the inverse system $\{W_\lambda \hookrightarrow U\}_{\lambda \in \Lambda}$ of all finitely presented affine immersions $W_\lambda \hookrightarrow U$ such that $x \in |W_\lambda|$. We claim that the inverse limit, i.e., the intersection, is $\mathcal{G}_x$.

Indeed, let $\pi \colon Z \to \underline{Z}$ denote the structure map of the gerbe. Then $\pi(x)$ is the intersection of its affine open neighborhoods $\underline{Z}_\alpha \subseteq \underline{Z}$. Thus $\mathcal{G}_x = \pi^{-1}(\operatorname{Spec} \kappa(\pi(x)))$ is the intersection of its relatively affine open neighborhoods $Z_\alpha = \pi^{-1}(\underline{Z}_\alpha)$, i.e., the open immersions $Z_\alpha \hookrightarrow Z$ are affine. Moreover, for a fixed α, we may pick an open quasi-compact substack $U_\alpha \subseteq U$ such that $Z_\alpha = Z \cap U_\alpha$. Since $Z_\alpha \hookrightarrow U_\alpha$ is a closed immersion, we may write $Z_\alpha \hookrightarrow U_\alpha$ as the intersection of closed immersions $Z_{\alpha\beta} \hookrightarrow U_\alpha$ of finite presentation [Ryd16]. For sufficiently large β, the immersion $Z_{\alpha\beta} \hookrightarrow U_\alpha \hookrightarrow U$ is affine, since the limit $Z_\alpha \hookrightarrow U_\alpha \hookrightarrow U$ is affine [Ryd15, Thm. C]. Thus $Z_{\alpha\beta} = W_\lambda$ for some $\lambda = \lambda(\alpha, \beta)$ for every α and every sufficiently large β. It follows that

$$\mathcal{G}_x \hookrightarrow \bigcap_{\lambda \in \Lambda} W_\lambda \hookrightarrow \bigcap_{\alpha} Z_\alpha = \mathcal{G}_x$$

and the result follows. ☐

3. Nisnevich dévissage

In this section, we consider Nisnevich coverings for quasi-separated algebraic stacks. For schemes, this goes back to the work of [Nis89] with the most famous applications due to [MV99]. In the setting of equivariant schemes this was considered in [HKØ15, §2]. It was also considered for Deligne–Mumford stacks in [KØ12, §§7-8]. The restriction to quasi-separated algebraic stacks is so that we can give an intuitive definition in terms of residual gerbes.

Definition 3.1. A morphism of quasi-separated algebraic stacks $p \colon W \to X$ is a *Nisnevich covering* if it is étale and for every $x \in |X|$, there exists an $w \in |W|$ such that $p(w) = x$ and the induced map of residual gerbes $\mathcal{G}_w \to \mathcal{G}_x$ is an isomorphism.

Nisnevich coverings are stable under composition and base change.

Example 3.2. Let X be a quasi-compact and quasi-separated scheme. Then there exists an affine scheme W and a Nisnevich covering $p \colon W \to X$. Indeed, taking $W = \coprod_{i=1}^n U_i$, where the $\{U_i\}$ form a finite affine open covering of X gives the claim. More generally, this holds for quasi-compact and quasi-separated algebraic spaces [RG71, Prop. 5.7.6].

Let $p \colon W \to X$ be a morphism of algebraic stacks. Recall that when p is not representable, then a section of p need not be a monomorphism. A *monomorphic splitting sequence* for p is a sequence of quasi-compact open immersions

$$\emptyset = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_r = X$$

such that p restricted to $X_i \setminus X_{i-1}$, when given the induced reduced structure, admits a monomorphic section for each $i = 1, \ldots, r$. In this situation, we say that p has a monomorphic splitting sequence of length r.

We have the following characterization of Nisnevich coverings, which is well-known for noetherian schemes [MV99, Lem. 3.1.5].

Proposition 3.3. *Let X be a quasi-compact and quasi-separated algebraic stack and let $p\colon W \to X$ be a quasi-separated étale morphism. Then p is a Nisnevich covering if and only if there exists a monomorphic splitting sequence for p.*

Proof. Let $x \in |X|$ be a point. Then there exists an immersion $Z_x \hookrightarrow X$ of finite presentation, such that $x \in |Z_x|$, and a monomorphic section of $p|_{Z_x}$. Indeed, there is a monomorphic section of $p|_{\mathcal{G}_x}$ which extends to a monomorphic section of $p|_{Z_x}$ by Lemma 2.1 and [Ryd15, Prop. B.2 (i) and B.3 (ii)].

The Z_x are constructible and we can thus cover X by a finite number of the Z_x's. We can thus filter X by a sequence of quasi-compact open substacks X_i such that $X_i \setminus X_{i-1}$ is contained in some Z_x. That is, we have obtained a monomorphic splitting sequence. ☐

The following lemma outlines the key benefits of the Nisnevich topology: it is generated by particularly simple coverings (cf. [MV99, Prop. 1.4]).

Lemma 3.4 (Nisnevich dévissage)**.** *Let S be a quasi-compact and quasi-separated algebraic stack and let $\mathbf{E} \subseteq \mathbf{Stack}_{\mathrm{fp},\text{ét}/S}$ be a full 2-subcategory containing all open immersions and closed under fiber products (e.g., one of the categories listed in the introduction). Let $\mathbf{D} \subseteq \mathbf{E}$ be a full 2-subcategory such that*

(N1) *if $(X' \to X) \in \mathbf{E}$ is an open immersion and $X \in \mathbf{D}$, then $X' \in \mathbf{D}$; and*

(N2) *if $(U \xrightarrow{i} X)$, $(X' \xrightarrow{f} X) \in \mathbf{E}$, where i is an open immersion and f is an isomorphism over $X \setminus U$, then $X \in \mathbf{D}$ whenever U, $X' \in \mathbf{D}$.*

If $p\colon W \to X$ is a Nisnevich covering in $\mathbf{E}$ and $W \in \mathbf{D}$, then $X \in \mathbf{D}$.

Proof. By Proposition 3.3, there is a sequence of quasi-compact open immersions:

$$\emptyset = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_r = X,$$

such that f restricted to $X_i \setminus X_{i-1}$, when given the induced reduced structure, admits a monomorphic section for $i = 1, \dots, r$. We will prove the result by induction on $r \geq 0$. If $r = 0$, then the result is trivial.

If $r > 0$, let $U = X_{r-1}$; then U admits a splitting sequence of length $r-1$. By the inductive hypothesis and (N1), we may thus assume that $U \in \mathbf{D}$. If $Z = (X \setminus U)_{\mathrm{red}}$, then the restriction of p to Z admits a section s, which is a quasi-compact open immersion. It follows that $X' = p^{-1}(U) \cup s(Z) = W \setminus (p^{-1}(Z) \setminus s(Z))$ is a quasi-compact open subset of W. Let $f : X' \to X$ be the induced morphism; then $X' \in \mathbf{D}$ and f is an isomorphism over $X \setminus U$. By (N2), the result follows. ☐

4. PRESENTATIONS OF ALGEBRAIC STACKS WITH FINITE STABILIZERS

The following theorem removes the separated diagonal assumption from [Hal16, Thm. B.5]. It will be crucial for the proofs of Theorems E and 5.1.

Theorem 4.1. *Let X be a quasi-compact and quasi-separated algebraic stack with quasi-finite diagonal. Then there exist morphisms of algebraic stacks*

$$V \xrightarrow{v} W \xrightarrow{p} X$$

such that

- *V is an affine scheme;*
- *v is finite, faithfully flat and of finite presentation; and*
- *p is a Nisnevich covering of finite presentation with separated diagonal.*

In addition,

(1) *if X has separated diagonal, then it can be arranged that p is representable and separated; and*
(2) *if X is Deligne–Mumford, then it can be arranged that v is étale.*

Proof. The proof is similar to [Ryd13, Prop. 6.11], [Ryd11a, Thms. 6.3 & 7.2] and [Hal16, Thm. B.5].

By [Ryd11a, Thm. 7.1], there is an affine scheme U and a representable, quasi-finite, faithfully flat and finitely presented morphism $u\colon U \to X$. The Hilbert stack $\underline{\mathrm{HS}}_{U/X} = \amalg_{d\geq 0}\mathscr{H}^d_{U/X} \to X$ parametrizing quasi-finite representable morphisms to U is algebraic and has quasi-affine—in particular, separated—diagonal [Ryd11b, Thm. 4.4]. Let $p\colon W = \underline{\mathrm{HS}}^{\text{ét}}_{U/X} \to X$ be the open substack of the Hilbert stack that parameterizes representable étale morphisms to U. Since u is flat, it is readily seen that $p\colon W \to X$ is étale.

We now prove that p is a Nisnevich covering. Let $x \in |X|$ be a point with residual gerbe $\mathcal{G}_x$. The restriction $u_x\colon U_x \to \mathcal{G}_x$ is finite and flat. Thus, the identity $U_x \to U_x$ corresponds to a section $\mathcal{G}_x \to W$. It is readily seen that this is a monomorphic section (e.g., by considering the open substack $H \subseteq W$ below).

After replacing W by a quasi-compact open subset containing the sections of a monomorphic splitting sequence (Proposition 3.3), we obtain a finitely presented Nisnevich covering $p\colon W \to X$. Let $v\colon V \to W$ be the universal family, which is finite (even étale if u is étale), flat and of finite presentation. Then there is a 2-commutative diagram

$$\begin{tikzcd}
V \arrow[r, "q"] \arrow[d, "v"'] & U \arrow[d, "u"] \\
W \arrow[r, "p"] & X,
\end{tikzcd}$$

where p and q are étale. After shrinking W, we may assume that v is surjective. Although p and q need neither be representable nor separated, we saw that p, and hence q, have separated diagonals. It follows that V has separated diagonal, and hence so has W [Ryd11a, Lem. A.4]. We may replace X by W and assume that X has separated diagonal.

When X has separated diagonal, the presentation u is separated. Consider the substack $H = \underline{\mathrm{Hilb}}^{\mathrm{open}}_{U/X} \subseteq W$ parameterizing open and closed immersions into U over X. In general H is not algebraic but since u is separated it is an open substack of W and $H \to S$ is representable and separated [Ryd11b, Thm. 4.1]. We may thus replace W with a quasi-compact open subset of H containing the sections. Then we obtain a commutative diagram as above where p and q are étale, representable and separated. By Zariski's Main Theorem [LMB, Thm. A.2], q is quasi-affine. By [Ryd13, Thm. 5.3], W has a coarse space $\pi\colon W \to W_{\mathrm{cs}}$ such that W_{cs} is a quasi-affine scheme and $\pi \circ v$ is affine (and integral). By Example 3.2, we may further reduce to the situation where W_{cs} is an affine scheme. Then V is affine and the result follows. □

Remark 4.2. A special case of (1) is when X has finite inertia. Then one can give an alternative proof of Theorem 4.1 using that X admits a coarse space $X \to X_{\mathrm{cs}}$ and that Nisnevich-locally on X_{cs}, we can find a finite flat presentation of X. Indeed, one immediately reduces to the case where X_{cs} is local henselian and then a quasi-finite flat presentation $U \to X$ splits as $U = V \amalg V'$ where $V \to X$ is finite and surjective.

5. HILBERT STACK OF STACKY POINTS

Let $f\colon X \to S$ be a morphism of algebraic stacks. Let $\underline{\mathrm{HS}}_{X/S}$ be the *Hilbert stack* of f. The Hilbert stack of f parameterizes quasi-finite and representable morphisms to X that are proper over the base. In [HR15b, HR14], it was proved that $\underline{\mathrm{HS}}_{X/S}$ was algebraic when f has quasi-finite and separated diagonal. The proof of this relies on the results of [HR14], whose methods are quite involved and may not be so familiar to the reader.

In this article, we will only need a small piece of $\underline{\mathrm{HS}}_{X/S}$: the open substack $\underline{\mathrm{HS}}^{\mathrm{qfb}}_{X/S}$ consisting of those families that are quasi-finite (though not necessarily representable) over the base. We will call this the *Hilbert stack of stacky points.* Using Nisnevich coverings, we will be able to deduce the algebraicity of the Hilbert stack of stacky points from the well-known algebraicity result in the case where f is separated, which is much easier (e.g, [Lie06], [Hal17, Thm. 9.1] and [HR15b, Thm. A(i)]).

Theorem 5.1. *If $f\colon X \to S$ is a morphism of algebraic stacks with quasi-compact and separated diagonal, then $\underline{\mathrm{HS}}^{\mathrm{qfb}}_{X/S}$ is an algebraic stack with quasi-affine diagonal over S. If f is locally of finite presentation (resp. is separated), then $\underline{\mathrm{HS}}^{\mathrm{qfb}}_{X/S}$ is locally of finite presentation (resp. has affine diagonal).*

To prove Theorem 5.1 we first prove a result on Weil restrictions.

Proposition 5.2. *Let $Z \to S$ be a quasi-finite, proper and flat morphism of finite presentation between quasi-separated algebraic stacks. If $U \to Z$ is a quasi-separated morphism with quasi-finite diagonal, then the Weil restriction $\mathbf{R}_{Z/S}(U) \to S$ is a quasi-separated algebraic stack. Moreover, if $U \to Z$ is*

1. *a Nisnevich covering; or*
2. *étale; or*
3. *representable; or*
4. *representable and separated; or*
5. *quasi-compact,*

then so too is $\mathbf{R}_{Z/S}(U) \to S$. If $U \to Z$ has separated diagonal, then $\mathbf{R}_{Z/S}(U) \to S$ has quasi-affine diagonal.

If $U \to Z$ has separated diagonal, it can be deduced that $\mathbf{R}_{Z/S}(U)$ is algebraic with quasi-affine diagonal using [HR15b, Thm. 2.3(vi)]. This relies on [HR14], however. We will avoid the reliance on [HR14] and the separated diagonal assumption when $Z \to S$ is quasi-finite using a simple bootstrapping process and Theorem 4.1.

Proof of Proposition 5.2. A standard argument shows that properties (2), (3), and (4) are preserved by taking Weil restrictions whenever the Weil restrictions in question exist, cf. [HR15b, Rem. 2.5]. To prove (1) when $\mathbf{R}_{Z/S}(U) \to S$ is already known to be a quasi-separated algebraic stack, we may replace S with a residual gerbe $\mathcal{G}_s$ for some point $s \in |S|$. Then $|Z|$ is finite and discrete. Thus, if $U \to Z$ is a Nisnevich covering, then $U \to Z$ has a monomorphic section. It follows that there is a monomorphic section $S \to \mathbf{R}_{Z/S}(U)$.

We make the following well-known observation: if $u\colon U_1 \to U_2$ is a morphism of algebraic stacks over Z, then the base change of $\mathbf{R}_{Z/S}(u)\colon \mathbf{R}_{Z/S}(U_1) \to \mathbf{R}_{Z/S}(U_2)$ along a morphism $T \to \mathbf{R}_{Z/S}(U_2)$, corresponding to a Z-morphism $Z \times_S T \to U_2$, is isomorphic to $\mathbf{R}_{Z\times_S T/T}((Z \times_S T) \times_{U_2} U_1)$. It follows that if P is a property of morphisms of algebraic stacks that is smooth-local on the target, then $\mathbf{R}_{Z/S}(u)$ is P if $\mathbf{R}_{Z/S}(U) \to S$ is P for all affine S and all $U \to Z$ satisfying P.

We next address the algebraicity. If $U \to Z$ is separated (resp. separated and representable), then $\mathbf{R}_{Z/S}(U) \to S$ is well-known to be algebraic with affine diagonal (resp. representable and separated), see [HR15b, Thm. 2.3(v)].

The algebraicity is smooth local on S, so we may assume that S is an affine scheme. Every section of $U \to Z$ factors through a quasi-compact open subset and Weil-restrictions of open substacks are open substacks, hence we may assume that U is quasi-compact. Theorem 4.1 implies that there is a Nisnevich covering $p\colon W \to U$ such that W has finite diagonal and $W \to U$ has separated diagonal. By the case already considered, $\mathbf{R}_{Z/S}(W) \to S$ is algebraic with affine diagonal. Consider the induced morphism $\mathbf{R}_{Z/S}(p)\colon \mathbf{R}_{Z/S}(W) \to \mathbf{R}_{Z/S}(U)$.

If $U \to Z$ has separated diagonal, then Theorem 4.1 even says that we can choose the Nisnevich covering $p\colon W \to U$ to be separated and representable. The separated case already considered and (1)–(4) now establishes that $\mathbf{R}_{Z/S}(p)$ is a representable and separated Nisnevich covering. Hence, $\mathbf{R}_{Z/S}(U) \to S$ is algebraic. To see that it has quasi-affine diagonal, we note that $\mathbf{R}_{Z/S}(U) \times_S \mathbf{R}_{Z/S}(U) \cong \mathbf{R}_{Z/S}(U \times_Z U)$. In particular, $\Delta_{\mathbf{R}_{Z/S}(U)} \simeq \mathbf{R}_{Z/S}(\Delta_{U/Z})$. Since $\Delta_{U/Z}$ is quasi-affine, $\mathbf{R}_{Z/S}(\Delta_{U/Z})$ is quasi-affine [HR15b, Thm 2.3(iii)].

If $U \to Z$ does not have separated diagonal, then $p\colon W \to U$ still has separated diagonal. Hence, by the cases already considered, $\mathbf{R}_{Z/S}(p)$ is algebraic and a Nisnevich étale covering. It follows that $\mathbf{R}_{Z/S}(U)$ is algebraic, but we still need to prove that it is quasi-separated. Repeating the argument above on separation conditions for $\mathbf{R}_{Z/S}(U) \to S$, the quasi-separatedness follows from (5).

It remains to show (5): the Weil restriction $R := \mathbf{R}_{Z/S}(U) \to S$ is quasi-compact if $U \to Z$ is quasi-compact. This claim is smooth local on S so we may assume that S is affine. Pick a quasi-finite flat presentation $Z' \to Z$ and let $Z'' = Z' \times_Z Z'$ and $Z''' = Z' \times_Z Z' \times_Z Z'$. To show that R is quasi-compact, we may replace S with a stratification. We may thus assume that $Z' \to S$ is finite. Then $R' := \mathbf{R}_{Z'/S}(U \times_Z Z') \to S$, $R'' := \mathbf{R}_{Z''/S}(U \times_Z Z'') \to S$ and $R''' := \mathbf{R}_{Z'''/S}(U \times_Z Z''') \to S$ are quasi-compact and quasi-separated algebraic stacks [Ryd11b, Prop. 3.8 (xiii) & (xix)]. If we define P (descent data without the descent condition) by the cartesian square

$$\begin{tikzcd}
R' \arrow[d, "(\pi_1^*, \pi_2^*)"'] \arrow[dr, phantom, "\square"] & P \arrow[l] \arrow[d] \\
R'' \times_S R'' & R'' \arrow[l, "\Delta"']
\end{tikzcd}$$

then there is a cartesian square

$$\begin{tikzcd}
P \arrow[d, "\tau"'] \arrow[dr, phantom, "\square"] & R \arrow[l] \arrow[d] \\
I_{R'''} & R''' \arrow[l, "e"']
\end{tikzcd}$$

by fppf descent [Ols07, Rmk. 4.4]. It follows that R is quasi-compact. ☐

We can now prove Theorem 5.1.

Proof of Theorem 5.1. We may assume that S is an affine scheme. If $X^{\mathrm{qf}} \subseteq X$ denotes the open substack where X has a quasi-finite diagonal, then it is clear that $\underline{\mathrm{HS}}^{\mathrm{qfb}}_{X^{\mathrm{qf}}/S} = \underline{\mathrm{HS}}^{\mathrm{qfb}}_{X/S}$; thus we may assume that X has quasi-finite and separated diagonal. Further standard reductions permit us to assume that X is also quasi-compact. By Theorem 4.1, there is a finitely presented, representable, and separated Nisnevich covering $p\colon W \to X$ such that W admits a finite flat and finitely presented covering by an affine scheme V. If X is separated, we instead let $W = X$. In either case, W has finite diagonal. By [HR15b, Thm. A(i)], $\underline{\mathrm{HS}}^{\mathrm{qfb}}_{W/S}$ is an algebraic stack with affine diagonal.

Let T be an affine scheme and let $(Z \to X \times_S T) \in \underline{\mathrm{HS}}^{\mathrm{qfb}}_{X/S}(T)$. It is well-known that the following diagram is 2-cartesian:

$$\begin{tikzcd}
\mathbf{R}_{Z/T}((W \times_S T) \times_{X \times_S T} Z) \arrow[r] \arrow[d] & T \arrow[d] \\
\underline{\mathrm{HS}}^{\mathrm{qfb}}_{W/S} \arrow[r] & \underline{\mathrm{HS}}^{\mathrm{qfb}}_{X/S},
\end{tikzcd}$$

and we conclude that $\underline{\mathrm{HS}}^{\mathrm{qfb}}_{W/S} \to \underline{\mathrm{HS}}^{\mathrm{qfb}}_{X/S}$ is a finitely presented, representable, and separated Nisnevich covering (Proposition 5.2). The theorem follows. □

Example 5.3. Theorem 5.1 is false if $X \to S$ has non-separated diagonal. This is similar to the main result of [LS08] (cf. [HR14]). For an explicit example, consider $S = \mathbb{A}^1_k$, where k is a field, and let $G = (\mathbb{Z}/2\mathbb{Z})_S$. Let $H \subseteq G$ be the étale subgroup scheme which is the complement of the non-trivial element lying over the origin in S. The quotient G/H is non-separated (it is just the line with the doubled origin). Let $X = B_S(G/H)$. Let $S_n = \mathrm{Spec}(k[x]/x^{n+1})$ and $\hat{S} = \mathrm{Spec}\, k[[x]]$. The natural map $(B_S G) \times_S S_n \to X \times_S S_n$ is representable (even an isomorphism), but there is no extension of this to a representable morphism $Y \to X \times_S \hat{S}$, where $Y \to \hat{S}$ is proper and flat.

Remark 5.4. If $X \to S$ is non-separated, then the natural object to consider is the 2-stack parameterizing not necessarily representable morphisms $Z \to X$ that are quasi-finite and flat over the base. This 2-stack ends up being algebraic because the proof of Theorem 5.1 holds verbatim. If $X \to S$ is flat and we restrict to the 2-substack parameterizing those $Z \to X$ that are also étale, then this is an étale 2-stack. In particular, it is an étale 2-gerbe over a 1-stack. Unfortunately, this 1-stack does not carry a universal family, which makes applying the result difficult. In particular, to prove dévissage results for morphisms with non-separated diagonals, it appears necessary to enter the world of higher stacks, cf. Remark 6.2.

6. NON-REPRESENTABLE PRESENTATIONS

The following theorem combines and extends [Ryd13, Prop. 6.11] and [Ryd11a, Thm. 6.3]. It makes crucial use of Theorem 5.1.

Theorem 6.1. *Let X be a quasi-compact and quasi-separated algebraic stack and let $u\colon U \to X$ be a quasi-finite and faithfully flat morphism of finite presentation with separated diagonal. Then there exists a commutative diagram of algebraic stacks*

$$\begin{tikzcd}
V \arrow[r, "q"] \arrow[d, "v"'] & U \arrow[d, "u"] \\
W \arrow[r, "p"] & X
\end{tikzcd}$$

such that

- *v is quasi-finite, proper and faithfully flat of finite presentation;*
- *p is a Nisnevich étale covering of finite presentation with separated diagonal; and*
- *q is an étale morphism of finite presentation with separated diagonal.*

In addition,

(1) *if u is representable, then it can be arranged that v is representable;*

(2) *if u is separated, then it can be arranged that p and q are separated and representable; and*
(3) *if u is étale, then it can be arranged that v is étale.*

Proof. Argue exactly as in the proof of the first part of Theorem 4.1. As before we take $W = \underline{\mathrm{HS}}^{\text{ét}}_{U/X}$, the open substack of the Hilbert stack $\underline{\mathrm{HS}}_{U/X}$ parameterizing étale morphisms to U. Since $U \to X$ is quasi-finite, $\underline{\mathrm{HS}}_{U/X} = \underline{\mathrm{HS}}^{\mathrm{qfb}}_{U/X}$ is algebraic with quasi-affine diagonal (Theorem 5.1). As before, it follows that $W \to X$ is étale with quasi-affine, hence separated, diagonal. If u is separated, we replace W with the open substack $\underline{\mathrm{Hilb}}^{\mathrm{open}}_{U/X}$ which is separated and representable over X. □

Remark 6.2. If u does not have separated diagonal in Theorem 6.1, then using the Hilbert 2-stack of Remark 5.4, we would arrive at the conclusion of the Theorem except that p and q need not have separated diagonals and are merely 2-representable, though v is still 1-representable. Here n-representable means represented by algebraic n-stacks. In particular, V and W are algebraic 2-stacks.

7. APPLICATIONS

In this section, we use non-representable étale dévissage to relax some separatedness conditions in the approximation results of [Ryd15] and the compact generation results of [HR17].

Lemma 7.1. *Let S be a quasi-compact and quasi-separated algebraic stack. Let X be a quasi-compact and quasi-separated algebraic stack over S and let $\pi\colon \mathscr{X} \to X$ be a proper fppf gerbe. Suppose $\mathscr{X} = \varprojlim_{\lambda\in\Lambda} \mathscr{X}_\lambda$ where $\mathscr{X}_\lambda$ are algebraic stacks of finite presentation over S and $g_\lambda\colon \mathscr{X} \to \mathscr{X}_\lambda$ are affine morphisms. Then for all sufficiently large λ, there is a commutative diagram*

$$\begin{tikzcd}
\mathscr{X} \arrow[r] \arrow[d, "\pi"'] \arrow[rr, bend left, "g_\lambda"] \arrow[dr, phantom, "\square"] & \mathscr{X}^\circ_\lambda \arrow[r, "i_\lambda"'] \arrow[d, "\pi_\lambda"] & \mathscr{X}_\lambda \\
X \arrow[r] & X^\circ_\lambda &
\end{tikzcd}$$

where i_λ is a finitely presented closed immersion, π_λ is a proper fppf gerbe and the square is cartesian. In particular, $X \to X^\circ_\lambda$ is affine and $X^\circ_\lambda \to S$ is of finite presentation.

Proof. The map π gives an exact sequence of group objects over $\mathscr{X}$

$$0 \to I_{\mathscr{X}/X} \to I_{\mathscr{X}/S} \to \pi^* I_{X/S}.$$

That π is an fppf gerbe of finite presentation implies that $I_{\mathscr{X}/X}$ is flat and of finite presentation. Conversely, given a flat subgroup $G \subseteq I_{\mathscr{X}/S}$ of finite presentation, there exists a *rigidification*: an algebraic stack $\mathscr{X} /\!\!/ G$ over S together with an fppf gerbe $\mathscr{X} \to \mathscr{X} /\!\!/ G$ of finite presentation such that the relative inertia is G [AOV08, Thm. A.1].

Let $G = I_{\mathscr{X}/X}$ and fix an index $\alpha \in \Lambda$. The inertia stack $I_{\mathscr{X}_\alpha/S}$ does not pullback to $I_{\mathscr{X}/S}$ but the canonical map $I_{\mathscr{X}/S} \to I_{\mathscr{X}_\alpha/S} \times_{\mathscr{X}_\alpha} \mathscr{X}$ is a closed subgroup stack. Since $G \to \mathscr{X}$ and $I_{\mathscr{X}_\alpha/S} \to \mathscr{X}_\alpha$ are of finite presentation, there is, by standard approximation methods [Ryd15, Props. B.2, B.3], an index $\lambda \geq \alpha$ and a subgroup $G_\lambda \hookrightarrow I_{\mathscr{X}_\alpha/S} \times_{\mathscr{X}_\alpha} \mathscr{X}_\lambda$ of finite presentation that pulls back to $G \hookrightarrow I_{\mathscr{X}_\alpha/S} \times_{\mathscr{X}_\alpha} \mathscr{X}$. After increasing λ, we may assume that $G_\lambda \to \mathscr{X}_\lambda$ is flat and proper [Ryd15, Prop. B.3].

We now address the problem that G_λ need not be a subgroup of $I_{\mathscr{X}_\lambda/S}$. Let $H_\lambda = G_\lambda \cap I_{\mathscr{X}_\lambda/S}$ as subgroups of $I_{\mathscr{X}_\alpha/S} \times_{\mathscr{X}_\alpha} \mathscr{X}_\lambda$. Then $H_\lambda \to G_\lambda$ is a finitely presented closed subgroup and $H_\lambda \times_{\mathscr{X}_\lambda} \mathscr{X} \to G_\lambda \times_{\mathscr{X}_\lambda} \mathscr{X}$ is an isomorphism. It follows that the Weil restriction $\mathscr{X}_\lambda^\circ := \mathbf{R}_{G_\lambda/\mathscr{X}_\lambda}(H_\lambda)$ is a finitely presented closed substack of $\mathscr{X}_\lambda$ and that $g_\lambda \colon \mathscr{X} \to \mathscr{X}_\lambda$ factors uniquely through $\mathscr{X}_\lambda^\circ$. Also note that after restricting to $\mathscr{X}_\lambda^\circ$, the closed subgroup $H_\lambda \to G_\lambda$ becomes an isomorphism. We thus have the subgroup $G_\lambda^\circ := G_\lambda|_{\mathscr{X}_\lambda^\circ} \to I_{\mathscr{X}_\lambda^\circ/S}$ which is proper and flat over $\mathscr{X}_\lambda^\circ$.

Let $X_\lambda^\circ = \mathscr{X}_\lambda^\circ /\!\!/ G_\lambda^\circ$. It remains to prove that we have a cartesian diagram. Since $\mathscr{X} \to X$ is initial among maps $\mathscr{X} \to Y$ such that $G \hookrightarrow I_{\mathscr{X}/S}$ factors through $I_{\mathscr{X}/Y} \hookrightarrow I_{\mathscr{X}/S}$, we have a map $X \to X_\lambda^\circ$. This induces a map between gerbes $\mathscr{X} \to \mathscr{X}_\lambda^\circ \times_{X_\lambda^\circ} X$ over X. This is a stabilizer-preserving morphism, i.e., $I_{\mathscr{X}/X} = G \to I_{\mathscr{X}_\lambda^\circ/X_\lambda^\circ} \times_{\mathscr{X}_\lambda^\circ} \mathscr{X} = G_\lambda^\circ \times_{\mathscr{X}_\lambda^\circ} \mathscr{X}$ is an isomorphism. But a stabilizer-preserving morphism between gerbes is an isomorphism. ☐

We can now remove most of the representability assumption in [Ryd15, Lemma. 7.9].

Proposition 7.2. *Let S be a pseudo-noetherian stack and let $X \to S$ be a morphism of algebraic stacks. Let $W \to X$ be an étale surjective morphism of finite presentation with separated diagonal (e.g., representable). If $W \to S$ can be approximated, then so can $X \to S$.*

Proof. We will apply étale dévissage (Theorem D′). Let $\mathbf{D} \subseteq \mathbf{E} = \mathbf{Stack}_{\mathrm{sep}_\Delta,\mathrm{fp},\mathrm{\acute{e}t}/S}$ be the full subcategory of morphisms $Y \to X$ such that $Y \to S$ is of strict approximation type or, equivalently, has an approximation [Ryd15, Prop. 4.8]. Then (D1) is satisfied by definition; (D2) for finite morphisms is [Ryd15, Prop. 2.12 (ii)] and (D3) is [Ryd15, Lem. 7.8]. It remains to prove (D2) for proper non-representable morphisms. Thus, let $Y' \to Y$ be a proper étale surjective morphism in $\mathbf{E}$. There is a canonical factorization $Y' \to Y'' \to Y$ where the first morphism is an étale gerbe and the second is finite étale. It is thus enough to prove (D2) when $Y' \to Y$ is a proper étale gerbe.

By assumption $Y' \to S$ has an approximation and can thus be written as $Y' = \varprojlim_\lambda Y'_\lambda$ where $Y'_\lambda \to S$ are of finite presentation and $Y' \to Y'_\lambda$ is affine for every λ. By Lemma 7.1 we have a cartesian diagram

$$\begin{tikzcd}
Y' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Y'^{\circ}_\lambda \arrow[d] \\
Y \arrow[r] & Y^\circ_\lambda .
\end{tikzcd}$$

of algebraic stacks over S where $Y \to Y_\lambda^\circ$ is affine and $Y_\lambda^\circ \to S$ is of finite presentation. Thus, $Y \to S$ has an approximation. ☐

In [Ryd15] it is shown that quasi-compact algebraic stacks with quasi-finite and *locally separated* diagonal can be approximated and are pseudo-noetherian. We can now remove the locally separatedness assumption.

Corollary 7.3. *Let X be a quasi-compact algebraic stack with quasi-finite and quasi-separated diagonal. Then $X \to \operatorname{Spec} \mathbb{Z}$ has an approximation. In particular, X is pseudo-noetherian.*

Proof. By Theorem 4.1, there is an étale surjective morphism $W \to X$ of finite presentation with separated diagonal (a Nisnevich cover) and a finite faithfully flat morphism $V \to W$ of finite presentation where V is an affine scheme. We conclude that W has an approximation by [Ryd15, Prop. 2.12 (ii)] and that X has an approximation by Proposition 7.2. ☐

We can also establish the following improvement of [HR17, Thm. A] in equicharacteristic 0, where it was proved for stacks with quasi-finite and separated diagonal.

Theorem 7.4. *Let X be a quasi-compact and quasi-separated Deligne–Mumford stack of* equicharacteristic 0*. Then the unbounded derived category $\mathsf{D}_{\mathrm{qc}}(X)$, of $\mathcal{O}_X$-modules with quasi-coherent cohomology, is compactly generated by a single perfect complex. Moreover, for every quasi-compact open subset $U \subseteq X$, there exists a compact perfect complex with support exactly $X \setminus U$.*

Proof. We apply Theorem E: let $\mathbf{D} \subseteq \mathbf{E} = \mathbf{Stack}_{\mathrm{sep}_{\Delta},\mathrm{fp},\acute{\mathrm{e}}\mathrm{t}/X}$ be the full subcategory consisting of those morphisms of Deligne–Mumford stacks $(W \to X)$, where for every quasi-compact open immersion $V \subseteq W$ we have that V satisfies the conclusion of the Theorem. This makes condition (I1) a triviality. Condition (I2) follows immediately from [HR17, Thm. A]. For Condition (I3) we use the theory developed in [HR17, §§5-6], with the following minor changes. In [HR17, Ex. 5.2], the working example throughout those sections, they take $\mathcal{D}$ to consist of representable and finitely presented morphisms to X; we will take $\mathcal{D} = \mathbf{E}$. The main difference is that $\mathcal{D}$ is now a 2-category, but the results go through without change. Since all morphisms of Deligne–Mumford stacks in equicharacteristic 0 are concentrated (combine [HR17, Lem. 2.5(2)] with [HR15a, Thm. C]), the resulting $(\mathcal{L}, \mathcal{D})$-presheaf of triangulated categories is admissible in the sense of [HR17, Defn. 6.1]; also see [HR17, Ex. 6.2] for further details and notations. Condition (I3) now follows from [HR17, Prop. 6.8]. ☐

Corollary 7.5. *If X is a noetherian Deligne–Mumford stack of equicharacteristic 0, then there is an equivalence of categories:*

$$\mathsf{D}(\mathsf{QCoh}(X)) \to \mathsf{D}_{\mathrm{qc}}(X).$$

Proof. Combine Theorem 7.4 with [HNR17, Thm. 1.2]. ☐

Remark 7.6. If $p \colon W \to X$ is a morphism of algebraic stacks and W has separated diagonal, then p has separated diagonal. This means that the étale presentations appearing in [AHR15, AHR14] always have separated diagonal.

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