ALGEBRAIC GROUPS AND COMPACT GENERATION OF THEIR DERIVED CATEGORIES OF REPRESENTATIONS

JACK HALL AND DAVID RYDH

Abstract. Let k be a field. We characterize the group schemes G over k, not necessarily affine, such that $\mathsf{D}_{\mathrm{qc}}(B_k G)$ is compactly generated. We also describe the algebraic stacks that have finite cohomological dimension in terms of their stabilizer groups.

Introduction

In this article we characterize two classes of group schemes over a field k:

(1) those with compactly generated derived categories of representations; and
(2) those with finite (Hochschild) cohomological dimension.

Compact generation. Let X be a quasi-compact and quasi-separated algebraic stack. Let $\mathsf{D}_{\mathrm{qc}}(X)$ be the unbounded derived category of lisse-étale $\mathcal{O}_X$-modules with quasi-coherent cohomology sheaves.

In [HR14], we showed that $\mathsf{D}_{\mathrm{qc}}(X)$ is compactly generated in many cases. This does not always hold, however. With Neeman, we considered $B_k\mathbb{G}_a$—the classifying stack of the additive group scheme over a field k—and proved that every compact object of $\mathsf{D}_{\mathrm{qc}}(B_k\mathbb{G}_a)$ is 0 if k has positive characteristic [HNR14, Prop. 3.1]. In particular, $\mathsf{D}_{\mathrm{qc}}(B_k\mathbb{G}_a)$ is not compactly generated.

If $\mathsf{D}_{\mathrm{qc}}(X)$ is compactly generated, then for every point $x\colon \operatorname{Spec} k \to X$ it follows that $\mathsf{D}_{\mathrm{qc}}(B_k G_x)$ is compactly generated, where G_x denotes the stabilizer group of x. It follows that the presence of a $\mathbb{G}_a$ in a stabilizer group of positive characteristic is an obstruction to compact generation [HNR14, Thm. 1.1]. We called such stacks *poorly stabilized*. Our first main result is that this obstruction is the only point-wise obstruction.

Theorem A. *Let k be a field, let G be a group scheme of finite type over k and let $\overline{G} = G \otimes_k \overline{k}$. Then $\mathsf{D}_{\mathrm{qc}}(B_k G)$ is compactly generated if and only if*

(1) *k has characteristic zero or*
(2) *k has positive characteristic and the reduced connected component $\overline{G}^0_{\mathrm{red}}$ is semi-abelian.*

Moreover, if $\mathsf{D}_{\mathrm{qc}}(B_k G)$ is compactly generated, then it is compactly generated by

(a) *a single perfect complex if and only if the affinization of $\overline{G}^0_{\mathrm{red}}$ is unipotent (e.g., if G is proper or unipotent); or*
(b) *the set of k-representations of G that have compact image in $\mathsf{D}_{\mathrm{qc}}(B_k G)$ when G is affine; or*
(c) *the set of irreducible k-representations of G when G is affine and k has characteristic zero or G is linearly reductive.*

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A group scheme is *semi-abelian* if it is an extension of an abelian variety by a torus (e.g., a torus or an abelian variety). Note that $\overline{G}^0_{\mathrm{red}}$ is semi-abelian precisely when there is no subgroup $\mathbb{G}_a \hookrightarrow \overline{G}$ [HNR14, Lem. 4.1]. The *affinization* of a group scheme G is the affine group scheme $\operatorname{Spec}\Gamma(G, \mathcal{O}_G)$, see [DG70, III.3.8].

Recall that the abelian category $\mathsf{QCoh}(B_kG)$ is naturally identified with the category $\mathsf{Rep}_k(G)$ of k-linear, locally finite representations of G. An *irreducible* k-representation of G is a simple object of the abelian category $\mathsf{Rep}_k(G)$. There is a natural functor

$$\Psi_{B_kG} \colon \mathsf{D}(\mathsf{Rep}_k(G)) = \mathsf{D}(\mathsf{QCoh}(B_kG)) \to \mathsf{D}_{\mathrm{qc}}(B_kG).$$

When G is affine and $\mathsf{D}_{\mathrm{qc}}(B_kG)$ is compactly generated, then Ψ_{B_kG} is an equivalence [HNR14, Thm. 1.2]. Conversely, if G is affine and $\mathsf{D}_{\mathrm{qc}}(B_kG)$ is not compactly generated, then G is poor (Theorem A) and Ψ_{B_kG} is not an equivalence [HNR14, Thm. 1.3]. If G is not affine, then Ψ_{B_kG} is not even full on bounded objects. Nonetheless, $\mathsf{D}_{\mathrm{qc}}(B_kG)$ remains preferable. For example, $\mathsf{D}_{\mathrm{qc}}(B_kG)$ is always left-complete, which is not true of $\mathsf{D}(\mathsf{QCoh}(B_kG))$; see [HNR14].

By Theorem A(c), if G is linearly reductive, then $\mathsf{D}_{\mathrm{qc}}(B_kG)$ is compactly generated by the finite-dimensional irreducible k-representations of G. Since $\mathsf{Rep}_k(G)$ is a semisimple abelian category, $\mathsf{Rep}_k(G)$ is generated by the finite-dimensional irreducible k-representations.

Theorem A(c) also implies that $\mathsf{D}_{\mathrm{qc}}(B_kG)$ is compactly generated by $\mathcal{O}_{B_kG}$ when G is unipotent and k has characteristic zero. We wish to point out, however, that the abelian category $\mathsf{Rep}_k(G)$ is not generated by the trivial one-dimensional representation [Gro13, Cor. 3.4]. This further emphasizes the benefits of the derived category $\mathsf{D}_{\mathrm{qc}}(B_kG)$ over the abelian category $\mathsf{Rep}_k(G)$.

Theorem A(c) cannot be extended to the situation where B_kG is not of finite cohomological dimension (e.g., it fails for $k = \overline{\mathbb{F}}_2$ and $G = (\mathbb{Z}/2\mathbb{Z})_k$). To prove Theorem A, we explicitly describe a set of generators (Remark 3.4).

Finite cohomological dimension. Let X be a quasi-compact and quasi-separated algebraic stack. An object of $\mathsf{D}_{\mathrm{qc}}(X)$ is *perfect* if it is smooth-locally isomorphic to a bounded complex of free $\mathcal{O}_X$-modules of finite rank. While every compact object of $\mathsf{D}_{\mathrm{qc}}(X)$ is perfect [HR14, Lem. 4.4 (1)], there exist non-compact perfect complexes (e.g., $\mathcal{O}_X$, where $X = B_{\mathbb{F}_2}(\mathbb{Z}/2\mathbb{Z})$). The following, however, are equivalent [HR14, Rem. 4.6]:

- every perfect object of $\mathsf{D}_{\mathrm{qc}}(X)$ is compact;
- the structure sheaf $\mathcal{O}_X$ is compact;
- there exists an integer d_0 such that for every quasi-coherent sheaf F on X, the cohomology groups $H^d(X, F)$ vanish for all $d > d_0$; and
- the derived global section functor $\mathsf{R}\Gamma \colon \mathsf{D}_{\mathrm{qc}}(X) \to \mathsf{D}(\mathsf{Ab})$ commutes with small coproducts.

We say that the stack X has *finite cohomological dimension* when it satisfies any of the conditions above.

In the relative situation, the cohomological dimension of a morphism depends in a subtle way on the separation properties of the target (see Remark 1.6). For this reason, in [HR14], we introduced the more robust notion of a *concentrated* morphism. In the absolute situation, these two notions coincide, and we will use them interchangeably.

If G is a group scheme over a field k, a basic question to consider is when its classifying stack B_kG is concentrated. In characteristic $p > 0$, the presence of unipotent subgroups of G (e.g., $\mathbb{Z}/p\mathbb{Z}$, $\boldsymbol{\alpha}_p$, or $\mathbb{G}_a$) is an immediate obstruction. This rules out all non-affine group schemes and GL_n, where $n > 1$. In characteristic zero, if G is affine, then its classifying stack is concentrated. It was surprising to us that in characteristic zero, there are non-affine group schemes whose classifying stack is concentrated. This follows from a recent result of Brion on the coherent cohomology of anti-affine group schemes [Bri13]. More precisely, we have the following theorem.

Theorem B. *Let k be a field, let G be a group scheme of finite type over k and let $\overline{G} = G \otimes_k \overline{k}$. Then $B_k G$ is concentrated if and only if*

1. *k has positive characteristic and $\overline{G}$ is affine and linearly reductive; or*
2. *k has characteristic zero and $\overline{G}$ is affine; or*
3. *k has characteristic zero and the anti-affine part G_{ant} of $\overline{G}$ is of the form $G_{\mathrm{ant}} = S \times_A E(A)$, where A is an abelian variety, $S \to A$ is an extension by a torus and $E(A) \to A$ is the universal vector extension.*

Finally, from Theorem B using stratifications and approximation techniques, we obtain a criterion for a stack to be concentrated.

Theorem C. *Let X be a quasi-compact and quasi-separated algebraic stack. Consider the following conditions:*

1. *X is concentrated.*
2. *Every residual gerbe $\mathcal{G}$ of X is concentrated.*
3. *For every point $x\colon \operatorname{Spec} k \to X$, the stabilizer group scheme G_x is as in Theorem B.*

Then (1) $\implies$ (2) $\iff$ (3). If X has affine stabilizer groups and either equal characteristic or finitely presented inertia, then (3) $\implies$ (1).

Theorem C generalizes a result of Drinfeld and Gaitsgory [DG13, Thm. 1.4.2]: in characteristic zero, every quasi-compact and quasi-separated algebraic stack with finitely presented inertia and affine stabilizers is concentrated. Our generalization is made possible by a recent approximation result of the second author [Ryd15a].

As an application of Theorem C and [HR14, Thm. C], we obtain the following variant of [HR14, Thm. B] in positive characteristic:

Theorem D. *Let X be an algebraic stack of equal characteristic. Suppose that there exists a faithfully flat, representable, separated and quasi-finite morphism $X' \to X$ of finite presentation such that X' has the resolution property and affine linearly reductive stabilizers. Then the unbounded derived category $\mathsf{D}_{\mathrm{qc}}(X)$ is compactly generated by a countable set of perfect complexes. In particular, this holds for every stack X of s-global type with linearly reductive stabilizers.*

Proof. Argue exactly as in the proof of [HR14, Thm. B] in [HR14, §9]: by [HR14, Ex. 8.9] and Theorem C, the stack X' is $\aleph_0$-crisp, hence so is X by [HR14, Thm. C]. □

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1. Cohomological dimension of classifying stacks

Let G be a group scheme of finite type over a field k. In this section, we give a complete classification of the groups G such that BG has finite cohomological dimension (Theorem B). In positive characteristic, these are the linearly reductive groups (Theorem 1.2). In characteristic zero, these are the affine groups as well as certain groups built up from the universal vector extension of an abelian variety (Theorem 1.4).

Definition 1.1. Let G be an *affine* group scheme over a field k of characteristic p. We say that G is

- *nice* if the connected component of the identity G^0 is of multiplicative type and the number of geometric components of G is not divisible by p; or
- *reductive* if the unipotent radical of $G_{\overline{k}}$ is trivial (G not necessarily connected); or
- *linearly reductive* if every finite dimensional representation of G is semi-simple, or equivalently, if $BG \to \operatorname{Spec} k$ has cohomological dimension zero.

Note that subgroups, quotients and extensions of nice group schemes are nice. Indeed, this follows from the corresponding fact for connected group schemes of multiplicative type [SGA3$_{\mathrm{II}}$, Exp. IX, Props. 8.1, 8.2]. Also note that if G is nice, then G^0 is a twisted form of $(\mathbb{G}_m)^n \times \boldsymbol{\mu}_{p^{r_1}} \times \cdots \times \boldsymbol{\mu}_{p^{r_m}}$ for some tuple of natural numbers $n, r_1, r_2, \ldots, r_m$.

If G is a group scheme of finite type over a field k, then there is always a smallest normal subgroup scheme G_{ant} such that G/G_{ant} is affine. The subgroup G_{ant} is anti-affine, that is, $\Gamma(G_{\mathrm{ant}}, \mathcal{O}_{G_{\mathrm{ant}}}) = k$. Anti-affine groups are always smooth, connected and commutative. Their structure has also been described by Brion [Bri09].

In positive characteristic, we have the following result, which is classical when G is smooth and affine.

Theorem 1.2 (Nagata's theorem)**.** *Let G be a group scheme of finite type over a field k. Consider the following conditions:*

1. *G is nice.*
2. *G is affine and linearly reductive.*
3. *BG has cohomological dimension* 0*.*
4. *BG has finite cohomological dimension.*

Then (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4)*. If k has positive characteristic, then all four conditions are equivalent.*

Proof. First, recall that group schemes of multiplicative type are linearly reductive. Moreover, a finite étale group scheme is linearly reductive if and only if the number of geometric components is prime to the characteristic p (by Maschke's Lemma and the fact that $\mathbb{Z}/p\mathbb{Z}$ is not linearly reductive).

(1) $\Longrightarrow$ (2): if G is nice, then G^0 and $\pi_0(G) = G/G^0$ are linearly reductive group schemes; thus, so is G (Lemma 1.3(2)).

(2) $\Longrightarrow$ (3): that an affine group scheme G is linearly reductive if and only if the classifying stack BG has cohomological dimension 0 is well-known.

Now, suppose that k has positive characteristic. That (2) $\Longrightarrow$ (1) when G is smooth is Nagata's theorem [Nag62]. That (2) $\Longrightarrow$ (1) in general is proved in [DG70, IV, §3, Thm. 3.6]. Let us briefly indicate how a similar argument proves that (4) $\Longrightarrow$ (1). Assume that BG has finite cohomological dimension. Then the same is true of BH for every subgroup H of G. In particular, there cannot be any subgroups of G isomorphic to $\mathbb{Z}/p\mathbb{Z}$ or $\boldsymbol{\alpha}_p$.

For the moment, assume that G is affine. If G is connected, then G is of multiplicative type since G has no subgroups isomorphic to $\boldsymbol{\alpha}_p$ [DG70, IV, §3, Lem. 3.7]. If G is disconnected, then the connected component G^0 has finite cohomological dimension and is thus of multiplicative type by the previous case. It follows that $\pi_0(G)$ has finite cohomological dimension (Lemma 1.3(3)). In particular, the rank has to be prime to p; hence G is nice.

Finally, suppose that G is not affine. Since we are in positive characteristic, G_{ant} is semi-abelian, i.e., the extension of an abelian variety A by a torus T [Bri09, Prop. 2.2]. In particular, the classifying stack BA has finite cohomological dimension. Indeed, $A = G_{\mathrm{ant}}/T$ and BT has cohomological dimension zero; then apply

Lemma 1.3(3). The subgroup scheme $A[p] \subseteq A$ of p-torsion points is finite of degree p^{2g}, where g is the dimension of A. By assumption, $A[p]$ has finite cohomological dimension, so $A[p]$ is of multiplicative type. But this is impossible: the Cartier dual is $A^\vee[p]$, which is not étale. □

Let $f\colon X \to Y$ be a quasi-compact and quasi-separated morphism of algebraic stacks. Define $\mathrm{cd}(f)$, the *cohomological dimension* of f, to be the least non-negative integer n such that $\mathsf{R}^d f_* M = 0$ for every $d > n$ and quasi-coherent sheaf M on X. If no such n exists, then we set $n = \infty$. We define the cohomological dimension of an algebraic stack X, $\mathrm{cd}(X)$, to be the non-negative integer $\mathrm{cd}(X \to \operatorname{Spec} \mathbb{Z})$.

The lemma that follows is a simple refinement of [Alp13, Prop. 12.17].

Lemma 1.3. *Let $H \hookrightarrow G$ be an inclusion of group schemes of finite type over a field k with quotient Q.*

(1) *Then $\mathrm{cd}(BH) \leq \mathrm{cd}(BG) + \mathrm{cd}(Q)$.*

In addition, if H is a normal subgroup scheme of G, then Q is a group scheme of finite type over k and the following holds:

(2) *$\mathrm{cd}(BG) \leq \mathrm{cd}(BH) + \mathrm{cd}(BQ)$; and*
(3) *if $\mathrm{cd}(BH) = 0$, then $\mathrm{cd}(BG) = \mathrm{cd}(BQ)$.*

Proof. Let $i\colon BH \to BG$ denote the induced morphism. For (1), by [HR14, Lem. 2.2(4)], $\mathrm{cd}(BH) \leq \mathrm{cd}(BG) + \mathrm{cd}(i)$. Also, the pull-back of i along the universal G-torsor is $Q \to \operatorname{Spec} k$. By [HR14, Lem. 2.2(2)], $\mathrm{cd}(i) \leq \mathrm{cd}(Q)$; the claim follows.

For (2), by [HR14, Lem. 2.2(4)], $\mathrm{cd}(BG) \leq \mathrm{cd}(BQ) + \mathrm{cd}(j)$, where $j\colon BG \to BQ$ is the induced morphism. Since $BH \to \operatorname{Spec} k$ is a pull-back of j, it follows that $\mathrm{cd}(j) \leq \mathrm{cd}(BH)$ [HR14, Lem. 2.2(2)]; the claim follows.

For (3), by (2), we know that $\mathrm{cd}(BG) \leq \mathrm{cd}(BQ)$. The reverse inequality follows from the observation that the underived adjunction map $\mathrm{Id}_{BQ} \to j_* j^*$ is an isomorphism and $\mathrm{cd}(j) = 0$. □

In characteristic zero, we have the following result.

Theorem 1.4. *Let G be a group scheme of finite type over a field k of characteristic zero. Then BG has finite cohomological dimension if and only if*

(1) *G is affine, i.e., G_{ant} is trivial; or*
(2) *G_{ant} is of the form $G_{\mathrm{ant}} = S \times_A E(A)$, where S is the extension of an abelian variety A by a torus and $E(A)$ is the universal vector extension of A.*

Proof. By Lemma 1.3(1)–(2), it is enough to treat the cases where G is either affine or anti-affine. If G is affine, then G is a closed subgroup of GL_n for some n. The induced morphism $BG \to B\mathrm{GL}_n$ is a GL_n/G-fibration. Since $\mathrm{cd}(B\mathrm{GL}_n) = 0$ in characteristic zero, it follows that $\mathrm{cd}(BG) \leq \mathrm{cd}(\mathrm{GL}_n/G)$ which is finite. In the anti-affine case, the result follows from Proposition 1.5. □

Proposition 1.5. *Let G be a non-trivial anti-affine group scheme of finite type over a field k. If k has characteristic zero and $G = S \times_A E(A)$, then BG has cohomological dimension zero. If not, then BG has infinite cohomological dimension.*

Proof. We have already seen that BG has infinite cohomological dimension in positive characteristic, so we may assume henceforth that k has characteristic zero.

By Chevalley's Theorem [Con02, Thm. 1.1], G is an extension of an abelian variety A by an affine connected group scheme G_{aff}. Since G is commutative, $G_{\mathrm{aff}} = T \times U$, where T is a torus and U is connected, unipotent and commutative; in particular, $U \cong (\mathbb{G}_a)^n$ for some n. Moreover, both the semi-abelian variety $S = G/U$ and the vector extension $E = G/T$ are anti-affine, and $G = S \times_A E$ [Bri09, Prop. 2.5]. Since T is linearly reductive, the cohomological dimension of $B(G/T)$ equals the cohomological dimension of BG (Lemma 1.3(3)). We may thus assume that $T = 0$, so that $G = E$ is an extension of A by U. Let g be the dimension of A and let n be the dimension of U.

Brion has calculated the coherent cohomology of G [Bri13, Prop. 4.3]:

$$H^*(G, \mathcal{O}_G) = \bigwedge^*(W^\vee),$$

where $W \subseteq H^1(A, \mathcal{O}_A)^\vee$ is a k-vector space of dimension $g - n$. If $g = n$, then G equals the universal vector extension $E(A)$ and G has no non-trivial cohomology.

We now proceed to calculate $H^*(BG, \mathcal{O}_{BG})$ via the Leray spectral sequence for the composition of $f\colon \operatorname{Spec} k \to BG$ and $\pi\colon BG \to \operatorname{Spec} k$. Some preliminary observations.

(1) Since G is anti-affine, every coherent sheaf on BG is a trivial vector bundle.
(2) If G was assumed to be an affine group scheme, then the natural functor $\Psi^+\colon \mathsf{D}^+(\mathsf{QCoh}(BG)) \to \mathsf{D}^+_{\mathrm{qc}}(BG)$ is an equivalence of categories and the derived functor $\mathsf{R}(f_{\text{lis-ét}})_*\colon \mathsf{D}^+_{\mathrm{qc}}(\operatorname{Spec} k) \to \mathsf{D}^+_{\mathrm{qc}}(BG)$ equals the composition of $\mathsf{R}(f_{\mathsf{QCoh}})_*\colon \mathsf{D}^+(\mathsf{Mod}(k)) \to \mathsf{D}^+(\mathsf{QCoh}(BG))$ with Ψ^+. When G is not affine, as in our case, both of these facts may fail.

First consider $\mathcal{H}^i\big(\mathsf{R}(f_{\text{lis-ét}})_* k\big) = \mathsf{R}^i f_* k \in \mathsf{QCoh}(BG)$. By flat base change, $f^*\mathsf{R}^i f_* k = H^i(G, \mathcal{O}_G)$, which is coherent of rank $d_i = \binom{g-n}{i}$. By the observation above, $\mathsf{R}^i f_* k$ is a trivial vector bundle of the same rank.

Consider the Leray spectral sequence:

$$E_2^{pq} = H^p(BG, \mathsf{R}^q f_* k) \Rightarrow E_\infty^{p+q} = H^{p+q}(\operatorname{Spec} k, k).$$

Of course, $H^n(\operatorname{Spec} k, k) = 0$, unless $n = 0$. Since $\mathsf{R}^q f_* k$ is trivial, we also have that $E_2^{pq} = H^p(BG, \mathcal{O}_{BG}) \otimes_k k^{d_q}$.

If $n = g$, then $E_2^{pq} = 0$ for all $q > 0$, so the spectral sequences degenerates and we deduce that $H^p(BG, \mathcal{O}_{BG}) = 0$ if $p > 0$. It follows that BG has cohomological dimension zero.

If $n < g$, then we claim that BG does not have finite cohomological dimension. In fact, suppose on the contrary that BG has finite cohomological dimension. Then E_2 is bounded with Euler characteristic zero, since $\sum_{i=0}^{g-n}(-1)^i d_i = 0$. This gives a contradiction since the Euler characteristic of E_∞ is one. □

Remark 1.6. The groups $G = S \times_A E(A)$ have quite curious properties. The classifying stack BG has cohomological dimension zero although G is not linearly reductive (for which we require G affine), showing that (3) does not always imply (2) in Theorem 1.2. Moreover, the presentation $f\colon \operatorname{Spec} k \to BG$ has cohomological dimension zero although f is not affine. This shows that in [HR14, Lem. 2.2 (6)], the assumption that Y has quasi-affine diagonal cannot be weakened beyond affine stabilizers. We also obtain an example of an extension $0 \to U \to E(A) \to A \to 0$ such that $\operatorname{cd}(BU) = g$, $\operatorname{cd}(BE(A)) = 0$ and $\operatorname{cd}(BA) = \infty$ for every $g \geq 1$. This shows that in Lemma 1.3, the cohomological dimension of BQ is not bounded by those of BG and BH unless $\operatorname{cd}(BH) = 0$.

Remark 1.7. In the proof of Proposition 1.5, we did not calculate the cohomology of BG for an anti-affine group scheme G. This can be done in characteristic zero as follows. Recall that G is the extension of the abelian variety A of dimension g by a commutative group $G_{\mathrm{aff}} = T \times U$, where T is a torus and $U \cong (\mathbb{G}_a)^n$ is a unipotent group of dimension $0 \leq n \leq g$. As before, we let $W \subseteq H^1(A, \mathcal{O}_A)^\vee$ be the k-vector space (of dimension $g - n$) corresponding to the vector extension

$0 \to U \to E \to A \to 0$. Then,

$$H^j(BG, \mathcal{O}_{BG}) = H^j(BE, \mathcal{O}_{BE}) = \begin{cases} \operatorname{Sym}^d(W^\vee) & \text{if } j = 2d \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

The first equality holds since BT has cohomological dimension zero. The second equality follows by induction on $g - n$. When $g - n = 0$ we saw that there is no higher cohomology. For $g - n > 0$, we consider the Leray spectral sequence for $BE' \to BE \to \operatorname{Spec} k$ where E' is a vector extension of A corresponding to a subspace $W' \subseteq W$ of dimension $g - n - 1$. An easy calculation gives the desired result.

In positive characteristic, $n = 0$ and $E = A$ and we expect that the cohomology is the same as above (with $W = H^1(A, \mathcal{O}_A)^\vee$). When $g = 1$, that is, when A is an elliptic curve, the Leray spectral sequence for $\operatorname{Spec} k \to BA \to \operatorname{Spec} k$ and an identical calculation as above confirms this.

2. Stabilizer groups and cohomological dimension

In this section, we generalize a result of Gaitsgory and Drinfeld [DG13, Thm. 1.4.2] on the cohomological dimension of noetherian algebraic stacks in characteristic zero with affine stabilizers. We extend their result to positive characteristic and also allow stacks with non-finitely presented inertia.

Theorem 2.1. *Let X be a quasi-compact and quasi-separated algebraic stack with affine stabilizers. If X is either*

(1) *a $\mathbb{Q}$-stack, or*
(2) *has nice stabilizers, or*
(3) *has nice stabilizers at points of positive characteristic and finitely presented inertia,*

then X is concentrated. In particular, this is the case if X is a tame Deligne–Mumford stack, or a tame Artin stack [AOV08].

Note that Theorems 1.2 and 1.4 give a partial converse to Theorem 2.1: if X is concentrated, then the stabilizer groups of X are either

(1) of positive characteristic and nice;
(2) of characteristic zero and affine; or
(3) of characteristic zero and extensions of an affine group by an anti-affine group of the form $S \times_A E(A)$.

Theorem C follows from Theorem 2.1 and this converse.

We will prove Theorem 2.1 by stratifying the stack into pieces that admit easy descriptions. For nice stabilizers, we need the following

Definition 2.2. A morphism of algebraic stacks $X \to Y$ is *nicely presented* if there exists:

(1) a constant finite group H such that $|H|$ is invertible over X,
(2) an H-torsor $E \to X$, and
(3) a $(\mathbb{G}_m)^n$-torsor $T \to E$ such that $T \to Y$ is quasi-affine.

We say that $X \to Y$ is *locally nicely presented* if $X \times_Y Y' \to Y'$ is nicely presented for some fppf-covering $Y' \to Y$.

Note that a locally nicely presented morphism has finite cohomological dimension. If Y has nice stabilizers (e.g., Y is a scheme) and $X \to Y$ is locally nicely presented, then X has nice stabilizers. The following lemma will also be useful.

Lemma 2.3. *Let G be a group algebraic space of finite presentation over a scheme S. If G has affine fibers, then the locus in S where the fibers are nice group schemes is constructible.*

Proof. Standard arguments reduce the situation to the following: S is noetherian and integral with generic point s and G is affine and flat over S. We may also replace S with S' for any dominant morphism $S' \to S$ of finite type. In particular, we may replace the residue field of the generic point with a finite field extension. Note that if the generic point has characteristic p, then S is an $\mathbb{F}_p$-scheme.

If the connected component of G_s is not of multiplicative type, then there exists, after a finite field extension, either a subgroup $\mathbb{G}_a \to G_s$ or a subgroup $\boldsymbol{\alpha}_p \to G_s$. By smearing out, there is an induced closed subgroup $(\mathbb{G}_a)_U \to G_U$ or $(\boldsymbol{\alpha}_p)_U \to G_U$, where U is open and dense in S; in particular, G_u is not nice for every $u \in U$.

If the connected component of G_s is of multiplicative type, there is, after a residue field extension, a sequence $0 \to T_s \to G_s \to H_s \to 0$ with T_s diagonalizable and H_s constant. We have T and H over S and we can spread out to an exact sequence over an open dense subscheme U of S that agrees with the pull back of G to U.

Let d be the order of H_s and p the characteristic of $\kappa(s)$. If G_s is nice, then $p \nmid d$. If p is zero, we may shrink U such that no point has characteristic dividing d. Thus G_u is nice for every $u \in U$. Conversely, if G_s is not nice, then $p|d$ and G_u is not nice for every $u \in U$. $\square$

Definition 2.4. Let X be an algebraic stack. A *finitely presented filtration* $(X_i)_{i=0}^r$ is a sequence of finitely presented closed substacks $\emptyset = X_0 \hookrightarrow X_1 \hookrightarrow \dots \hookrightarrow X_r \hookrightarrow X$ such that $|X_r| = |X|$.

Remark 2.5. If X is quasi-compact and quasi-separated with inertia of finite presentation (e.g., X noetherian), then there exists a finitely presented filtration of X with strata that are gerbes. In the noetherian case, this is immediate from generic flatness and [LMB, Prop. 10.8]. For the general case, see [Ryd15a, Cor. 8.4]. Moreover, by Lemma 2.3, if X has affine stabilizers as in Theorem B(1) or (2), then X has a stratification by gerbes such that each stratum is either of equal characteristic 0 or nice.

On a quasi-compact and quasi-separated algebraic stack, every quasi-coherent sheaf is a direct limit of its finitely generated quasi-coherent subsheaves. This is well-known for noetherian algebraic stacks [LMB, Prop. 15.4]. The general case was recently settled by the second author [Ryd15a].

Proposition 2.6. *Let X be a quasi-compact and quasi-separated algebraic stack. Then*

(1) *X has* affine *stabilizers if and only if there exists a finitely presented filtration $(X_i)_{i=0}^r$, positive integers $n_1, n_2, \dots, n_r$ and* quasi-affine *morphisms $X_i \setminus X_{i-1} \to B\mathrm{GL}_{n_i,\mathbb{Z}}$ for every $i = 1, \dots, r$; and*
(2) *X has* nice *stabilizers if and only if there exists a finitely presented filtration $(X_i)_{i=0}^r$, affine schemes S_i of finite presentation over* $\operatorname{Spec} \mathbb{Z}$ *and* locally nicely presented *morphisms $X_i \setminus X_{i-1} \to S_i$ for every $i = 1, \dots, r$.*

Proof. The conditions are clearly sufficient. To prove that they are necessary, first assume that X is an fppf-gerbe over an algebraically closed field k. Then $X = BG$, where G is an affine (resp. nice) group scheme. If G is affine, then there is a quasi-affine morphism to some $B\mathrm{GL}_{n,k}$ [Tot04, Lem. 3.1]. If G is nice, then $BG^0 \to BG$ is an $H = \pi_0(G)$-torsor. Since G^0 is diagonalizable, there is a $(\mathbb{G}_m)^n$-torsor $(\mathbb{G}_{m,k})^{n-r} \to BG^0$. Thus $BG \to \operatorname{Spec} k \to \operatorname{Spec} \mathbb{Z}$ is nicely presented.

If k is not algebraically closed, then, by approximation, the situation above holds after passing to a finite field extension k'/k. If the stabilizer of X is affine, then X has the resolution property [HR14, Rmk. 7.2] and hence there is a quasi-affine morphism $X \to B\mathrm{GL}_{n,k}$. In this case, let $S = B\mathrm{GL}_{n,\mathbb{Z}}$. If the stabilizer of X is nice, then $X \to \operatorname{Spec} k$ is at least locally nicely presented. By approximating $\operatorname{Spec} k \to \operatorname{Spec} \mathbb{Z}$, we obtain a finitely presented affine scheme $S \to \operatorname{Spec} \mathbb{Z}$ such that $X \to \operatorname{Spec} k \to S$ is locally nicely presented.

If X is any quasi-separated algebraic stack, then for every point $x \in |X|$ there is an immersion $Z \hookrightarrow X$ such that Z is an fppf-gerbe over an affine integral scheme $\underline{Z}$ and the residual gerbe $\mathcal{G}_x \to \operatorname{Spec} \kappa(x)$ is the generic fiber of $Z \to \underline{Z}$ [Ryd11, Thm. B.2]. In particular, $\mathcal{G}_x$ is the inverse limit of open neighborhoods $U \subseteq Z$ of x such that $U \to Z$ is affine. By [Ryd15b, Thm. C], there exists an open neighborhood $x \in U \subseteq Z$ and a morphism $U \to S$ that is quasi-affine (resp. locally nicely presented).

We may write the quasi-compact immersion $U \hookrightarrow Z \hookrightarrow X$ as a closed immersion $U \hookrightarrow V$ in some quasi-compact open substack $V \subseteq X$. Since V is quasi-compact and quasi-separated, we may express $U \hookrightarrow V$ as an inverse limit of finitely presented closed immersions $U_\lambda \hookrightarrow V$. Since S is of finite presentation, there is a morphism $U_\lambda \to S$ for sufficiently large λ. After increasing λ, the morphism $U_\lambda \to S$ becomes quasi-affine (resp. locally nicely presented) by [Ryd15b, Thm. C]. Let $U_x = U_\lambda$.

For every $x \in |X|$ proceed as above and choose a locally closed finitely presented immersion $U_x \hookrightarrow X$ with $x \in |U_x|$. As the substacks U_x are constructible, it follows by quasi-compactness that a finite number of the U_x's cover X and we easily obtain a stratification and filtration as claimed, cf. [Ryd11, Pf. of Prop. 4.4]. □

The following lemma will be useful.

Lemma 2.7 ([DG13, 2.3.2])**.** *Let X be a quasi-compact and quasi-separated algebraic stack. If $i\colon Z \hookrightarrow X$ is a finitely presented closed immersion with complement $j\colon U \hookrightarrow X$, then*

$$\operatorname{cd}(X) \leq \max\{\operatorname{cd}(U), \operatorname{cd}(Z) + \operatorname{cd}(j) + 1\}.$$

Proof. Let I denote the ideal sheaf defining Z in X. Let F be a quasi-coherent sheaf on X. Consider the adjunction map $F \to \mathsf{R}j_*j^*F$ and let C denote the cone. Then $j^*C = 0$ and C is supported in degrees $\leq \operatorname{cd}(j)$. Since $H^d(\mathsf{R}\Gamma\mathsf{R}j_*j^*F) = H^d(U, j^*F) = 0$ for $d > \operatorname{cd}(U)$, it is enough to show that $H^d(X,G) = 0$ if G is a quasi-coherent sheaf such that $j^*G = 0$ and $d > \operatorname{cd}(Z)$. After writing G as a direct limit of its finitely generated subsheaves, we may further assume that G is finitely generated. Then $I^nG = 0$ for sufficiently large n and one easily proves that $H^d(X,G) = 0$ by induction on n. □

We now prove the main result of this section.

Proof of Theorem 2.1. We first treat (1) and (2). Choose a filtration as in Proposition 2.6(1) or (2). In characteristic zero, $B\mathrm{GL}_n$ has cohomological dimension zero and quasi-affine morphisms have finite cohomological dimension. In arbitrary characteristic, locally nicely presented morphisms have finite cohomological dimension. Indeed, BH and $B(\mathbb{G}_m)^n$ have cohomological dimension zero. Thus, the Theorem follows from Lemma 2.7. For (3), we may choose a filtration as in Remark 2.5. Then the result follows from Lemma 2.7 and the cases (1) and (2) already proved. □

There are several other applications of the structure result of Proposition 2.6. An immediate corollary is that the locus of points where the stabilizers are affine (resp. nice) is ind-constructible. This is false for "linearly reductive": the locus with linearly reductive stabilizers in $B\mathrm{GL}_{n,\mathbb{Z}}$, for $n \geq 2$, is the subset $B\mathrm{GL}_{n,\mathbb{Q}}$ which is not ind-constructible. Another corollary is the following approximation result.

Theorem 2.8. *Let S be a quasi-compact algebraic stack and let $X = \varprojlim_\lambda X_\lambda$ be an inverse limit of quasi-compact and quasi-separated morphisms of algebraic stacks $X_\lambda \to S$ with affine transition maps. Then X has affine (resp. nice) stabilizers if and only if X_λ has affine (resp. nice) stabilizers for sufficiently large λ.*

Proof. The question is fppf-local on S, so we can assume that S is affine. Note that if $X \to Y$ is affine and Y has affine (resp. nice) stabilizers, then so has X. The result now follows from Proposition 2.6 and [Ryd15b, Thm. C]. □

Thus if X_λ is of equal characteristic and has affine stabilizer groups, then $X \to S$ has finite cohomological dimension if and only if $X_\lambda \to S$ has finite cohomological dimension for sufficiently large λ. The example $X = B\mathrm{GL}_{2,\mathbb{Q}} = \varprojlim_m B\mathrm{GL}_{2,\mathbb{Z}[\frac{1}{m}]}$ shows that this is false in mixed characteristic.

3. Compact generation of classifying stacks

In this section, we prove Theorem A on the compact generation of classifying stacks. The following three lemmas will be useful.

Lemma 3.1. *Let $F\colon \mathcal{T} \to \mathcal{S}$ be a triangulated functor between triangulated categories that are closed under small coproducts. Assume that F admits a conservative right adjoint G that preserves small coproducts. If $\mathcal{T}$ is compactly generated by a set T, then $\mathcal{S}$ is compactly generated by the set $F(T) = \{F(t) : t \in T\}$.*

Proof. By [Nee96, Thm. 5.1 "⇒"], $F(T) \subseteq \mathcal{S}^c$. Thus, it remains to prove that the set $F(T)$ is generating. If $s \in \mathcal{S}$ is non-zero, then $G(s)$ is non-zero. It follows that there is a non-zero map $t \to G(s)[n]$ for some $t \in T$ and $n \in \mathbb{Z}$. By adjunction, there is a non-zero map $F(t) \to s[n]$, and we have the claim. □

Lemma 3.2. *Let $\pi\colon X' \to X$ be a proper and faithfully flat morphism of noetherian algebraic stacks. Assume that either π is finite or a torsor for a smooth group scheme. If a set T compactly generates $\mathsf{D}_{\mathrm{qc}}(X')$, then the set $\{\mathsf{R}\pi_* P : P \in T\}$ compactly generates $\mathsf{D}_{\mathrm{qc}}(X)$.*

Proof. By [HR14, Ex. 6.5] and Proposition A.1, in both cases $\mathsf{R}\pi_*$ is D_{qc}-quasiperfect with respect to open immersions (see [HR14, Defn. 6.4]) and its right adjoint $\pi^!$ is conservative. The claim now follows from Lemma 3.1. □

Lemma 3.3. *Let k be a field and let $1 \to K \to G \to H \to 1$ be a short exact sequence of group schemes of finite type over k. Let $p\colon BG \to BH$ be the induced morphism. Assume that either*

(1) *$\mathsf{D}_{\mathrm{qc}}(BK)$ is compactly generated by $\mathcal{O}_{BK}$, or*
(2) *$K \subseteq G_{\mathrm{ant}}$ and $\mathrm{cd}(BK) = 0$.*

Then $\mathsf{R}p_*\colon \mathsf{D}_{\mathrm{qc}}(BG) \to \mathsf{D}_{\mathrm{qc}}(BH)$ is concentrated and conservative.

Proof. For (1), the pull back of p along the universal H-torsor is the morphism $p'\colon BK \to \operatorname{Spec} k$. Since $\mathsf{D}_{\mathrm{qc}}(BK)$ is compactly generated by $\mathcal{O}_{BK}$, it follows that BK is concentrated. By [HR14, Lem. 2.5(2)], p is concentrated. To prove that $\mathsf{R}p_*$ is conservative, by [HR14, Thm. 2.6], it remains to prove that $\mathsf{R}p'_*$ is conservative. If $M \in \mathsf{D}_{\mathrm{qc}}(BK)$ is non-zero, then by assumption there is a non-zero map $\mathcal{O}_{BK}[n] \to M$ for some integer n. Since $\mathsf{L}p'^*\mathcal{O}_{\operatorname{Spec} k} \simeq \mathcal{O}_{BK}$, by adjunction, there is a non-zero map $\mathcal{O}_{\operatorname{Spec} k}[n] \to \mathsf{R}p'_* M$. The claim follows.

For (2), by [HR14, Lem. 2.2(2) & 2.5(2)], $\mathrm{cd}(p) = 0$ and p is concentrated. Thus, if $M \in \mathsf{D}_{\mathrm{qc}}(BG)$ and $i \in \mathbb{Z}$, then $\mathcal{H}^i(\mathsf{R}p_* M) = p_*\mathcal{H}^i(M)$. So to establish that $\mathsf{R}p_*$

is conservative, it remains to prove that the functor $p_*\colon \mathsf{QCoh}(BG) \to \mathsf{QCoh}(BH)$ is conservative. Let $q\colon BG \to B(G/G_{\mathrm{ant}})$ be the natural morphism. Then q factors as $BG \xrightarrow{p} BH \to B(G/G_{\mathrm{ant}})$. Smooth-locally q is the morphism $BG_{\mathrm{ant}} \to \operatorname{Spec} k$, and $\mathsf{QCoh}(BG_{\mathrm{ant}}) \to \mathsf{QCoh}(\operatorname{Spec} k)$ is an equivalence [Bri09, Lem. 1.1]. By descent, it follows that q_* is conservative. Hence, p_* is conservative. The result follows. □

Proof of Theorem A. If k has positive characteristic and $\overline{G}^0_{\mathrm{red}}$ is not semi-abelian, then B_kG is poorly stabilized [HNR14, Lem. 4.1], so $\mathsf{D}_{\mathrm{qc}}(B_kG)$ is not compactly generated [HNR14, Thm. 1.1]. Conversely, assume either that k has characteristic zero or that $\overline{G}^0_{\mathrm{red}}$ is semi-abelian.

Let G^0 be the connected component of G. Then $BG^0 \to BG$ is finite and faithfully flat. By Lemma 3.2, we may assume that $G = G_0$. By Lemma 3.2, we may always pass to finite extensions of the ground field k. In particular, we may assume that G_{red} is a smooth group scheme. Similarly, since $BG_{\mathrm{red}} \to BG$ is finite and faithfully flat, we may replace G with G_{red}. Hence, we may assume that G is smooth and connected.

By Chevalley's Theorem [Con02, Thm. 1.1], we may (after passing to a finite extension of k) write G as an extension of an abelian variety A by a smooth connected affine group G_{aff}. By assumption, G_{aff} is a torus in positive characteristic. In particular, BG_{aff} is concentrated, has affine diagonal and the resolution property; thus $\mathsf{D}_{\mathrm{qc}}(BG_{\mathrm{aff}})$ is compactly generated by a set of compact vector bundles [HR14, Prop. 8.4]. Since the induced map $f\colon BG_{\mathrm{aff}} \to BG$ is an A-torsor, $\mathsf{D}_{\mathrm{qc}}(BG)$ is compactly generated (Lemma 3.2). Note that this also establishes (b).

For (c), let $M \in \mathsf{D}_{\mathrm{qc}}(B_kG)$ and suppose that $M \not\simeq 0$. By (b), there exists a non-zero map $V[n] \to M$, where V is a finite-dimensional k-representation of G. Let $L \subseteq V$ be an irreducible k-subrepresentation of G. If the composition $L[n] \to V[n] \to M$ is zero, then there is an induced non-zero map $(V/L)[n] \to M$. Since V is finite-dimensional, we must eventually arrive at the situation where there is a non-zero map $L[n] \to M$, where L is irreducible. Finally, B_kG has finite cohomological dimension (Theorem B), so L is compact [HR14, Rem. 4.6].

It remains to address (a). Suppose that $\mathsf{D}_{\mathrm{qc}}(BG)$ is compactly generated by a single perfect complex. Then so too is $\mathsf{D}_{\mathrm{qc}}(B\overline{G}^0_{\mathrm{red}})$. Assume that $k = \overline{k}$ and $G = \overline{G}^0_{\mathrm{red}}$; in particular, G is smooth and connected and k is perfect. To derive a contradiction, we assume that G/G_{ant}—the affinization of G—is not unipotent. By Chevalley's Theorem [Con02, Thm. 1.1], G is an extension of an abelian variety A by a connected smooth affine group G_{aff}. The exact sequence of [Bri09, Prop. 3.1(i)] quickly implies that the induced map $G_{\mathrm{aff}} \to G/G_{\mathrm{ant}}$ is surjective. In particular, G_{aff} is not unipotent; moreover, there is a subgroup $\mathbb{G}_m \subseteq G_{\mathrm{aff}}$ such that the induced map $\mathbb{G}_m \to G/G_{\mathrm{ant}}$ has kernel μ_n for some n. Since G/G_{ant} is affine and $\mathbb{G}_m$ is linearly reductive, it follows that the induced morphism $\phi\colon B(\mathbb{G}_m/\mu_n) \to B(G/G_{\mathrm{ant}})$ is affine; in particular, the functor $\mathsf{R}\phi_*$ is conservative.

Let $\mathcal{L}$ be the standard representation of $\mathbb{G}_m$. Then for every integer r, a brief calculation using that $\mathsf{R}\phi_*$ is conservative proves that $\mathsf{R}q_*(\mathcal{L}^{\otimes rn}) \neq 0$, where q is the composition $B\mathbb{G}_m \to B(\mathbb{G}_m/\mu_n) \xrightarrow{\phi} B(G/G_{\mathrm{ant}})$. If $\mathsf{D}_{\mathrm{qc}}(BG)$ is compactly generated by a single perfect complex P, then for every integer r there exist integers m_r and non-zero maps $l_r\colon P \to \mathsf{R}\psi_*(\mathcal{L}^{\otimes rn})[m_r]$, where $\psi\colon B\mathbb{G}_m \to BG_{\mathrm{aff}} \to BG$ is the induced map; indeed, $\mathsf{R}q_*$ is conservative so $\mathsf{R}\psi_*(\mathcal{L}^{\otimes rn}) \neq 0$ for every r. By adjunction, there are non-zero maps $\mathsf{L}\psi^*P \to \mathcal{L}^{\otimes rn}[m_r]$, for every r. That is,

$$\mathrm{Hom}_{\mathcal{O}_{B\mathbb{G}_m}}(\mathsf{L}\psi^*P, \mathcal{L}^{\otimes rn}[m_r]) = \mathrm{Hom}_{\mathcal{O}_{B\mathbb{G}_m}}(\psi^*\mathcal{H}^{m_r}(P), \mathcal{L}^{\otimes rn})$$

is non-zero for every integer r. But $\mathsf{L}\psi^*P$ is perfect, so there are only finitely many non-zero $\mathcal{H}^i(P)$ and only a finite number of the representations $\mathcal{L}^{\otimes rn}$ appear in $\psi^*\mathcal{H}^i(P)$. Hence, we have a contradiction, so the affinization of $\overline{G}^0_{\mathrm{red}}$ is unipotent.

Conversely, suppose that the affinization of $\overline{G}^0_{\mathrm{red}}$ is unipotent. By Lemma 3.2 and arguing as before, after passing to a finite extension of k, we may assume that $G = G^0_{\mathrm{red}}$ and that the affinization G/G_{ant} is unipotent. Passing to a further finite extension of k, by Chevalley's Theorem [Con02, Thm. 1.1], we may assume that G (resp. G_{ant}) is an extension of an abelian scheme A (resp. A') by a connected smooth affine group G_{aff} (resp. G'_{aff}). Note that if k has positive characteristic, then since $\mathsf{D}_{\mathrm{qc}}(BG)$ is compactly generated, it follows by what we have already established that G_{aff} has no unipotent elements; in particular, since $G_{\mathrm{aff}} \to G/G_{\mathrm{ant}}$ is surjective (arguing as above) and G/G_{ant} is unipotent, it follows that G/G_{ant} is trivial.

By [Bri09, Prop. 3.1(ii)] we have that $G'_{\mathrm{aff}} \subseteq G_{\mathrm{aff}}$. Since G'_{aff} is smooth, affine, connected and commutative, it follows that $G'_{\mathrm{aff}} = T \times U$, where T is a torus and U is connected and unipotent [Bri09, (2.5)]. Note that from the above, if k has positive characteristic, then $G'_{\mathrm{aff}} = T$. By assumption, G is connected; thus, $G_{\mathrm{ant}} \subseteq Z(G)$ [DG70, Cor. III.3.8.3]. In particular, T is a normal subgroup of both G_{aff} and G. By Lemmas 3.1 and 3.3(2), it suffices to prove that $\mathsf{D}_{\mathrm{qc}}(G/T)$ is compactly generated by a single perfect complex.

We have exact sequences

$$1 \longrightarrow G_{\mathrm{aff}}/T \longrightarrow G/T \longrightarrow A \longrightarrow 1$$

$$1 \longrightarrow U \longrightarrow G_{\mathrm{aff}}/T \longrightarrow G_{\mathrm{aff}}/G'_{\mathrm{aff}} \longrightarrow 1.$$

The kernel of the surjective map $G_{\mathrm{aff}}/G'_{\mathrm{aff}} \twoheadrightarrow G/G_{\mathrm{ant}}$ is finite by [Bri09, Prop. 3.1(ii)]. By assumption G/G_{ant} is unipotent and $G_{\mathrm{aff}}/G'_{\mathrm{aff}}$ is connected and smooth; hence $G_{\mathrm{aff}}/G'_{\mathrm{aff}}$ and G_{aff}/T are unipotent. Note that in positive characteristic $G_{\mathrm{aff}}/T = 0$.

We know that $\mathsf{D}_{\mathrm{qc}}(BA)$ is compactly generated by a single perfect complex (Lemma 3.2). In characteristic zero, since G_{aff}/T is unipotent, we have also established that $\mathsf{D}_{\mathrm{qc}}(B(G_{\mathrm{aff}}/T))$ is compactly generated by the structure sheaf in (c). Hence, by Lemmas 3.1 and 3.3(1), we have that $\mathsf{D}_{\mathrm{qc}}(B(G/T))$ is compactly generated and the result follows. □

Remark 3.4. In characteristic zero, the proof of Theorem A shows that if G^0 fits in an exact sequence of group schemes $0 \to U \to G^0 \to A \to 0$, where U is unipotent, then $\mathsf{D}_{\mathrm{qc}}(BG)$ is compactly generated by the perfect complex $\mathsf{R}\pi_*\mathcal{O}_{BU}$, where $\pi\colon BU \to BG$ is the induced morphism.

Corollary 3.5. *Let k be a field. Let $\mathcal{G}$ be a quasi-compact and quasi-separated fppf gerbe over* $\operatorname{Spec} k$*. The derived category $\mathsf{D}_{\mathrm{qc}}(\mathcal{G})$ is compactly generated if and only if $\mathcal{G}$ is not poorly stabilized.*

Proof. If $\mathcal{G}$ is poorly stabilized, then $\mathsf{D}_{\mathrm{qc}}(\mathcal{G})$ is not compactly generated [HNR14, Thm. 1.1]. Conversely, Lemma 3.2 permits us to reduce to the situation where $\mathcal{G}$ is neutral. The result now follows from Theorem A. □

More generally, we have the following.

Theorem 3.6. *Let S be a scheme and let $G \to S$ be a flat group scheme of finite presentation. Let X be a quasi-compact algebraic stack over S with quasi-finite and separated diagonal and let $\mathcal{G} \to X$ be a G-gerbe. Assume that either*

(1) *S is the spectrum of a field k and G is not poor, that is, either S has characteristic zero or $\overline{G}^0_{\mathrm{red}}$ is semi-abelian; or*
(2) *S is arbitrary and $G \to S$ is of multiplicative type.*

Then $\mathcal{G}$ is $\aleph_0$-crisp (and 1-crisp if $G \to S$ is proper). In particular, $\mathsf{D}_{\mathrm{qc}}(\mathcal{G})$ is compactly generated.

Proof. The question is local on X with respect to quasi-finite faithfully flat morphisms of finite presentation [HR14, Thm. C]. We may thus assume that X is affine and that $\mathcal{G} \to X$ is a trivial G-gerbe, that is, $\mathcal{G} \simeq X \times_S BG$. We may also replace S by a quasi-finite flat cover and in the first case assume that G^0_{red} is a group scheme and in the second case assume that $G \to S$ is diagonalizable.

In the second case $X \times_S BG$ is concentrated, has affine diagonal and has the resolution property. It is thus $\aleph_0$-crisp [HR14, Prop. 8.4].

In the first case, we may, after further base change, apply Chevalley's theorem and write G^0_{red} as an extension of an abelian variety A/k by a smooth connected affine group G_{aff} (a torus in positive characteristic). The stack $X \times_k BG_{\mathrm{aff}}$ is $\aleph_0$-crisp as in the previous case (1-crisp if G is proper). The morphism $X \times_k BG_{\mathrm{aff}} \to X \times_k BG^0_{\mathrm{red}}$ is a torsor under A, hence Proposition A.1 and [HR14, Prop. 6.6] applies. Hence, $X \times_k BG^0_{\mathrm{red}}$ is $\aleph_0$-crisp. Finally, since $BG^0_{\mathrm{red}} \to BG$ is finite and flat, $X \times_k BG$ is $\aleph_0$-crisp by [HR14, Thm. C]. ☐

Appendix A. Grothendieck duality for smooth and representable morphisms of algebraic stacks

In this Appendix we prove a variant of [Nir08, Prop. 1.20] that was necessary for this paper. The difficult parts of the following Proposition, for schemes, are well-known [Con00, Thm. 4.3.1].

Recall that a morphism of algebraic stacks $X \to Y$ is *schematic* (or strongly representable) if for every scheme Y' and morphism $Y' \to Y$, the pull-back $X \times_Y Y'$ is a scheme. We say that $X \to Y$ is *locally schematic* if there exists a faithfully flat morphism $Y' \to Y$, locally of finite presentation, such that $X \times_Y Y'$ is a scheme. In particular, if S is a scheme, $G \to S$ is a group *scheme*, Y is an S-stack and $X \to Y$ is a G-torsor, then $X \to Y$ is locally schematic (but perhaps not schematic).

Proposition A.1. *Let $f\colon X \to Y$ be a proper, smooth, and locally schematic morphism of noetherian algebraic stacks of relative dimension n. Let $f^!\colon \mathsf{D}_{\mathrm{qc}}(Y) \to \mathsf{D}_{\mathrm{qc}}(X)$ be the functor $\omega_f[n] \otimes_{\mathcal{O}_X} \mathsf{L}f^*(-)$, where $\omega_f = \wedge^n \Omega_f$.*

(1) *There is a* trace morphism *$\gamma_f\colon \mathsf{R}^n f_*\omega_f \to \mathcal{O}_Y$ that is compatible with locally noetherian base change on Y.*
(2) *The trace morphism induces a natural transformation $\mathrm{Tr}_f\colon \mathsf{R}f_*f^! \to \mathrm{Id}$, which is compatible with locally noetherian base change and gives rise to a sheafified duality quasi-isomorphism whenever $M \in \mathsf{D}_{\mathrm{qc}}(X)$ and $N \in \mathsf{D}_{\mathrm{qc}}(Y)$:*
$$J_{f,M,N}\colon \mathsf{R}f_*\mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(M, f^!N) \to \mathsf{R}\mathcal{H}om_{\mathcal{O}_Y}(\mathsf{R}f_*M, N).$$
In particular, $f^!$ is a right adjoint to $\mathsf{R}f_*\colon \mathsf{D}_{\mathrm{qc}}(X) \to \mathsf{D}_{\mathrm{qc}}(Y)$.

Proof. For the moment, assume that f is a morphism of schemes. By [Con00, Cor. 3.6.6], there is a *trace morphism* $\gamma_f\colon \mathsf{R}^n f_*\omega_f \to \mathcal{O}_Y$ that is compatible with locally noetherian base change on Y. For $N \in \mathsf{D}_{\mathrm{qc}}(Y)$, there is also an induced morphism, which we denote as $\mathrm{Tr}_f(N)$:
$$\mathsf{R}f_*f^!N \simeq (\mathsf{R}f_*\omega_f)[n] \otimes^{\mathsf{L}}_{\mathcal{O}_Y} N \to (\mathsf{R}^n f_*\omega_f) \otimes^{\mathsf{L}}_{\mathcal{O}_Y} N \xrightarrow{\gamma_f \otimes \mathrm{Id}} N,$$
where the first isomorphism is the Projection Formula [Nee96, Prop. 5.3] and the second morphism is given by the truncation map $\tau_{\geq 0}$—using that $\mathsf{R}f_*$ has cohomological dimension n. Tor-independent base change (e.g., [HR14, Cor. 4.13]) shows

that the morphism $\mathrm{Tr}_f(N)$ is natural and compatible with locally noetherian base change and induces a sheafified duality morphism:

$$J_{f,M,N}\colon \mathsf{R}f_*\mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(M, f^!N) \to \mathsf{R}\mathcal{H}om_{\mathcal{O}_Y}(\mathsf{R}f_*M, N),$$

where $M \in \mathsf{D}_{\mathrm{qc}}(X)$ and $N \in \mathsf{D}_{\mathrm{qc}}(Y)$. The morphism $J_{f,M,N}$ is a quasi-isomorphism whenever $M \in \mathsf{D}^b_{\mathsf{Coh}}(X)$ and $N \in \mathsf{D}^b_{\mathrm{qc}}(Y)$ [Con00, Thm. 4.3.1].

Returning to the general case, we note that by hypothesis, there is a noetherian scheme U and a smooth and surjective morphism $p\colon U \to Y$ such that in the 2-cartesian square of algebraic stacks:

$$\begin{tikzcd} X_U \arrow[r, "p_X"] \arrow[d, "f_U"'] & X \arrow[d, "f"] \\ U \arrow[r, "p"] & Y, \end{tikzcd}$$

the morphism f_U is a proper and smooth morphism of relative dimension n of noetherian schemes. Let $R = U \times_Y U$, which is a noetherian algebraic space. Let $\tilde{R} \to R$ be an étale surjection, where $\tilde{R}$ is a noetherian scheme. Let s_1 and s_2 denote the two morphisms $\tilde{R} \to R \to U$ and let $f_{\tilde{R}}\colon X_{\tilde{R}} \to \tilde{R}$ denote the pullback of f along $p \circ s_1\colon \tilde{R} \to Y$. By the above, there are trace morphisms γ_{f_U} and $\gamma_{f_{\tilde{R}}}$ that are compatible with locally noetherian base change. In particular, for $i = 1$ and $i = 2$ the following diagram commutes:

$$\begin{tikzcd} s_i^*\mathsf{R}^n(f_U)_*\omega_{f_U} \arrow[r, "\sim"] \arrow[d, "s_i^*\gamma_{f_U}"'] & \mathsf{R}^n(f_{\tilde{R}})_*\omega_{f_{\tilde{R}}} \arrow[d, "\gamma_{f_{\tilde{R}}}"] \\ s_i^*\mathcal{O}_U \arrow[r, "\sim"] & \mathcal{O}_{\tilde{R}} \end{tikzcd}$$

By smooth descent, there is a uniquely induced morphism $\gamma_f\colon \mathsf{R}^n f_*\omega_f \to \mathcal{O}_Y$ such that the following diagram commutes:

$$\begin{tikzcd} p^*\mathsf{R}^n f_*\omega_f \arrow[r, "\sim"] \arrow[d, "p^*\gamma_f"'] & \mathsf{R}^n(f_U)_*\omega_{f_U} \arrow[d, "\gamma_{f_U}"] \\ p^*\mathcal{O}_Y \arrow[r, "\sim"] & \mathcal{O}_U. \end{tikzcd}$$

Now the morphism f is quasi-compact, quasi-separated, and representable—whence concentrated [HR14, Lem. 2.5 (3)]. By the Projection Formula [HR14, Cor. 4.12], there is a natural quasi-isomorphism for each $N \in \mathsf{D}_{\mathrm{qc}}(Y)$:

$$(\mathsf{R}f_*\omega_f[n]) \otimes^{\mathsf{L}}_{\mathcal{O}_Y} N \simeq \mathsf{R}f_*f^!N.$$

Since f is also proper, flat, and representable with fibers of relative dimension $\leq n$ it follows that $\mathsf{R}f_*\omega_f \in \mathsf{D}^{[0,n]}_{\mathsf{Coh}}(Y)$. Inverting the quasi-isomorphism above and truncating, we obtain a natural morphism:

$$\mathsf{R}f_*f^!N \simeq (\mathsf{R}f_*\omega_f)[n] \otimes^{\mathsf{L}}_{\mathcal{O}_Y} N \to (\mathsf{R}^n f_*\omega_f) \otimes^{\mathsf{L}}_{\mathcal{O}_Y} N \xrightarrow{\gamma_f \otimes \mathrm{Id}} N,$$

which we denote as $\mathrm{Tr}_f(N)$. If $M \in \mathsf{D}_{\mathrm{qc}}(X)$ and $N \in \mathsf{D}_{\mathrm{qc}}(Y)$, let

$$\begin{aligned} A(M,N) &= \mathsf{R}f_*\mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(M, f^!N), \\ B(M,N) &= \mathsf{R}\mathcal{H}om_{\mathcal{O}_Y}(\mathsf{R}f_*M, N), \text{ and} \\ J_{f,M,N}&\colon A(M,N) \to B(M,N), \end{aligned}$$

where $J_{f,M,N}$ is the sheafified duality morphism induced by $\mathsf{R}f_*f^!N \to N$. Furthermore, there is a natural isomorphism of functors:

$$p^*\mathsf{R}f_*f^! \simeq \mathsf{R}(f_U)_*p_X^*f^! \simeq \mathsf{R}(f_U)_*(f_U)^!p^*,$$

and the following diagram is readily observed to commute for each $N \in \mathsf{D}_{\mathrm{qc}}(Y)$:

$$\begin{tikzcd} p^*\mathsf{R}f_*f^!N \arrow[r, "p^*\mathrm{Tr}_f(N)"] \arrow[d] & p^*N \arrow[d, equal] \\ \mathsf{R}(f_U)_*(f_U)^!p^*N \arrow[r, "\mathrm{Tr}_{f_U}(p^*N)"] & p^*N. \end{tikzcd}$$

We have already seen that $J_{f_U,M,N}$ is a quasi-isomorphism whenever $M \in \mathsf{D}^b_{\mathsf{Coh}}(X_U)$ and $N \in \mathsf{D}^b_{\mathrm{qc}}(U)$. Thus, by tor-independent base change and the commutativity of the diagram above, the morphism $J_{f,M,N}$ is a quasi-isomorphism whenever $M \in \mathsf{D}^b_{\mathsf{Coh}}(X)$ and $N \in \mathsf{D}^b_{\mathrm{qc}}(Y)$.

It remains to prove that $J_{f,M,N}$ is a quasi-isomorphism for all $M \in \mathsf{D}_{\mathrm{qc}}(X)$ and all $N \in \mathsf{D}_{\mathrm{qc}}(Y)$. By [HNR14, Thm. B.1], $\mathsf{D}_{\mathrm{qc}}(X)$ and $\mathsf{D}_{\mathrm{qc}}(Y)$ are left-complete triangulated categories. Thus, we have distinguished triangles:

$$N \to \prod_{k\le 0} \tau^{\ge k} N \to \prod_{k\le 0} \tau^{\ge k} N \quad \text{and} \quad f^!N \to \prod_{k\le 0} \tau^{\ge k} f^!N \to \prod_{k\le 0} \tau^{\ge k} f^!N,$$

where the first maps are the canonical ones and the second maps are $1-$shift. Since $f^![-n]$ is t-exact, we also have a distinguished triangle:

$$f^!N \to \prod_{k\le 0} f^!\tau^{\ge k} N \to \prod_{k\le 0} f^!\tau^{\ge k} N.$$

Hence we have a natural morphism of distinguished triangles:

$$\begin{tikzcd} A(M,N) \arrow[r] \arrow[d, "J_{f,M,N}"'] & \prod_{k\le 0} A(M,\tau^{\ge k}N) \arrow[r] \arrow[d, "(J_{f,M,\tau^{\ge k}N})"'] & \prod_{k\le 0} A(M,\tau^{\ge k}N) \arrow[d, "(J_{f,M,\tau^{\ge k}N})"'] \\ B(M,N) \arrow[r] & \prod_{k\le 0} B(M,\tau^{\ge k}N) \arrow[r] & \prod_{k\le 0} B(M,\tau^{\ge k}N). \end{tikzcd}$$

Since f has cohomological dimension $\le n$, it follows that there are natural quasi-isomorphisms for every pair of integers k and p:

$$\begin{aligned} \tau^{\le p} A(M,\tau^{\ge k}N) &= \tau^{\le p}\mathsf{R}f_*\mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(M, f^!\tau^{\ge k}N) \\ &\simeq \tau^{\le p}\mathsf{R}f_*\tau^{\le p}\mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(M, \tau^{\ge k-n}f^!N) \\ &\simeq \tau^{\le p}\mathsf{R}f_*\mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(\tau^{\ge k-n-p}M, f^!\tau^{\ge k}N) \\ &= \tau^{\le p} A(\tau^{\ge k-n-p}M, \tau^{\ge k}N) \\ \tau^{\le p} B(M,\tau^{\ge k}N) &= \tau^{\le p}\mathsf{R}\mathcal{H}om_{\mathcal{O}_Y}(\mathsf{R}f_*M, \tau^{\ge k}N) \\ &\simeq \tau^{\le p}\mathsf{R}\mathcal{H}om_{\mathcal{O}_Y}(\tau^{\ge k-p}\mathsf{R}f_*M, \tau^{\ge k}N) \\ &\simeq \tau^{\le p}\mathsf{R}\mathcal{H}om_{\mathcal{O}_Y}(\mathsf{R}f_*(\tau^{\ge k-n-p}M), \tau^{\ge k}N) \\ &= \tau^{\le p} B(\tau^{\ge k-n-p}M, \tau^{\ge k}N). \end{aligned} \tag{A.1}$$

Thus it is enough to establish that $J_{f,M,N}$ is a quasi-isomorphism when $M \in \mathsf{D}^+_{\mathrm{qc}}(X)$ and $N \in \mathsf{D}^+_{\mathrm{qc}}(Y)$. A similar argument, but this time using the homotopy colimit $\bigoplus_{k\ge 0} \tau^{\le k}M \to \bigoplus_{k\ge 0} \tau^{\le k}M \to M$ (cf., [LO08, Lem. 4.3.2]), further permits a reduction to the situation where $M \in \mathsf{D}^b_{\mathrm{qc}}(X)$ and $N \in \mathsf{D}^b_{\mathrm{qc}}(Y)$.

For the remainder of the proof we fix $N \in \mathsf{D}^b_{\mathrm{qc}}(Y)$. Let F_N be the functor $A(-,N)$ and let G_N be the functor $B(-,N)$, both regarded as contravariant triangulated functors from $\mathsf{D}_{\mathrm{qc}}(X)$ to $\mathsf{D}_{\mathrm{qc}}(Y)$. Since N is bounded below, the functors

F_N and G_N are bounded below (A.1) and $J_{f,-,N}$ induces a natural transformation $F_N \to G_N$.

Let $C \subseteq \mathsf{QCoh}(X)$ be the collection of objects of the form $\bigoplus_{i\in I} L_i$, where $L_i \in \mathsf{Coh}(X)$ and I is a set. Recall that $J_{f,L,N}$ is a quasi-isomorphism whenever $L \in \mathsf{Coh}(X)$ and $N \in \mathsf{D}^b_{\mathrm{qc}}(Y)$. Since F_N and G_N both send coproducts to products, it follows that $J_{f,\oplus L_i,N} = \prod J_{f,L_i,N}$, so $J_{f,L,N}$ is also a quasi-isomorphism whenever $L = \bigoplus L_i \in C$.

Every $M \in \mathsf{QCoh}(X)$ is a quotient of some object of C [LMB, Prop. 15.4]. By standard "way-out" arguments (e.g., [Lip09, Compl. 1.11.3.1]) it now follows that $J_{f,M,N}$ is a quasi-isomorphism for all $M \in \mathsf{D}^-_{\mathrm{qc}}(X)$, and the result follows. □

Remark A.2. Note that if A is an abelian variety and $\pi\colon BA \to \operatorname{Spec} k$ is the classifying stack, then $\mathsf{R}\pi_*\colon \mathsf{D}_{\mathrm{qc}}(BA) \to \mathsf{D}(\mathsf{Mod}(k))$ does not admit a right adjoint. In fact, BA is not concentrated (see Section 1), so $\mathsf{R}\pi_*$ does not preserve small coproducts; thus, cannot be a left adjoint.

References

[Alp13] J. Alper, *Good moduli spaces for Artin stacks*, Ann. Inst. Fourier (Grenoble) **63** (2013), no. 6, 2349–2402.

[AOV08] D. Abramovich, M. Olsson, and A. Vistoli, *Tame stacks in positive characteristic*, Ann. Inst. Fourier (Grenoble) **58** (2008), no. 4, 1057–1091.

[Bri09] M. Brion, *Anti-affine algebraic groups*, J. Algebra **321** (2009), no. 3, 934–952.

[Bri13] M. Brion, *The Coherent Cohomology Ring of an Algebraic Group*, Algebr. Represent. Theory **16** (2013), no. 5, 1449–1467.

[Con00] B. Conrad, *Grothendieck duality and base change*, Lecture Notes in Mathematics, vol. 1750, Springer-Verlag, Berlin, 2000.

[Con02] B. Conrad, *A modern proof of Chevalley's theorem on algebraic groups*, J. Ramanujan Math. Soc. **17** (2002), no. 1, 1–18.

[DG70] M. Demazure and P. Gabriel, *Groupes algébriques. Tome I: Géométrie algébrique, généralités, groupes commutatifs*, Masson & Cie, Éditeur, Paris, 1970, Avec un appendice *Corps de classes local* par Michiel Hazewinkel.

[DG13] V. Drinfeld and D. Gaitsgory, *On some finiteness questions for algebraic stacks*, Geom. Funct. Anal. **23** (2013), no. 1, 149–294.

[Gro13] P. Gross, *Tensor generators on schemes and stacks*, preprint, June 2013, arXiv:1306.5418.

[HNR14] J. Hall, A. Neeman, and D. Rydh, *One positive and two negative results for derived categories of algebraic stacks*, preprint, May 2014, arXiv:1405.1888v2.

[HR14] J. Hall and D. Rydh, *Perfect complexes on algebraic stacks*, preprint, May 2014, arXiv:1405.1887v2.

[Lip09] J. Lipman, *Notes on derived functors and Grothendieck duality*, Foundations of Grothendieck duality for diagrams of schemes, Lecture Notes in Math., vol. 1960, Springer, Berlin, 2009, pp. 1–259.

[LMB] G. Laumon and L. Moret-Bailly, *Champs algébriques*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge., vol. 39, Springer-Verlag, Berlin, 2000.

[LO08] Y. Laszlo and M. Olsson, *The six operations for sheaves on Artin stacks. I. Finite coefficients*, Publ. Math. Inst. Hautes Études Sci. (2008), no. 107, 109–168.

[Nag62] M. Nagata, *Complete reducibility of rational representations of a matric group.*, J. Math. Kyoto Univ. **1** (1961/1962), 87–99.

[Nee96] A. Neeman, *The Grothendieck duality theorem via Bousfield's techniques and Brown representability*, J. Amer. Math. Soc. **9** (1996), no. 1, 205–236.

[Nir08] F. Nironi, *Grothendieck Duality for Deligne-Mumford Stacks*, preprint, November 2008, arXiv:0811.1955.

[Ryd11] D. Rydh, *Étale dévissage, descent and pushouts of stacks*, J. Algebra **331** (2011), 194–223.

[Ryd15a] D. Rydh, *Approximation of sheaves on algebraic stacks*, Int. Math. Res. Not. (2015), Advance Access. http://dx.doi.org/10.1093/imrn/rnv142.

[Ryd15b] D. Rydh, *Noetherian approximation of algebraic spaces and stacks*, J. Algebra **422** (2015), 105–147.

[$\mathrm{SGA3_{II}}$] *Schémas en groupes. II: Groupes de type multiplicatif, et structure des schémas en groupes généraux*, Séminaire de Géométrie Algébrique du Bois Marie 1962/64 (SGA 3).

Dirigé par M. Demazure et A. Grothendieck. Lecture Notes in Mathematics, Vol. 152, Springer-Verlag, Berlin, 1970.

[Tot04] B. Totaro, *The resolution property for schemes and stacks*, J. Reine Angew. Math. **577** (2004), 1–22.

Mathematical Sciences Institute, The Australian National University, Acton ACT 2601, Australia

E-mail address: jack.hall@anu.edu.au

KTH Royal Institute of Technology, Department of Mathematics, SE-100 44 Stockholm, Sweden

E-mail address: dary@math.kth.se