Frequency Localising Basis Functions for Wide-band System Identification:
A Condition Number Bound for Output Error Systems

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Abstract— **Identifying the parameters of a system possessing a large dynamic range presents a number of difficulties for a least squares type algorithm. Foremost, is that of obtaining reliable parameter estimates from a poor or ill-conditioned least squares formulation. In this paper we examine properties of a technique that utilises frequency localising basis functions to ensure that ill-conditioning is reduced when solving for the parameter estimates. Specifically, we obtain a bound on the condition number of the least squares problem which is independent of the frequency range for a particular class of models. We also present an example, utilising real data, which demonstrates the potential of the technique when applied to large dynamic range systems.**

I. INTRODUCTION

In system identification one can perform parameter estimation in either the time or frequency domain [5, 7, 14, 17, 11, 12]. Several advantages exist for the case of frequency domain identification over that of identification in the time domain [10, 15]. These advantages include: the ease of noise reduction (only frequencies where excitation is provided are used in the estimation procedure), data reduction (through the use of a non-parametric model of the system in the frequency domain), the ease of combining data from different experiments and model validation (periodic excitation provides a good frequency response model at the excitation frequencies). In fact, frequency domain identification is usually preferred over that of the time domain for the modelling of continuous time systems, particulary when the excitation signal is periodic and a parametric model of a plant or process is required.

A linear single-input-single-output (SISO) dynamic system can be characterised by a transfer function involving the ratio of two polynomials. Levy [9] proposed a technique involving the use of a linear least squares (LS) estimation algorithm, for use with experimentally obtained frequency domain data, to estimate the coefficients of these polynomials. It is well known that the normal matrix utilised in this type of estimator is sensitive to the dynamics and bandwidth of the system and can lead to ill-conditioning. This ill-conditioning, of the LS normal matrix, has been well studied [6]. The ramification of poor conditioning is that it typically manifests itself as poor or erroneous estimates of the system parameters.

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There has been a substantial amount of work undertaken to improve the conditioning of the LS problem for a variety of reasons in the system identification area. To overcome a poor low frequency fit [16] proposed an iterative method utilising a weighting obtained from the previous estimate. To improve the numerical properties of the LS problem [2] used frequency scaling and [1, 2] utilised orthogonal polynomials, such as Tchebychev polynomials. Orthonormal basis functions have been shown to provide perfect conditioning for the LS problem under specific conditions [19, 13], i.e. for a white input spectrum. It has also been shown [13] that the orthonormal basis functions exhibit some degree of robustness with respect to spectral colouring of the input. However, as demonstrated in [20], there is still significant ill-conditioning associated with all these methods when considering systems with a large dynamic range and more general inputs.

It should be noted that, in general, it is not good practice to explicitly form and then invert the normal matrix to solve the least squares problem. Several techniques exist which allow one to find the LS estimate without inverting the normal matrix [6, 7]. However, if the conditioning of the regressor matrix is very poor, then these other methods still encounter difficulties in obtaining a reliable solution.

Other recent work [4] has proposed an algorithm that uses S-K iterations [16] and frequency scaling as well as orthogonal polynomials as basis functions. Evidence provided in [4], based on real experimental data, shows that the condition number for the examples, is relatively low.

A technique proposed in [20], utilises particular basis functions, which are aimed specifically at improving the numerical properties of the LS problem in transfer function estimation over a large dynamic range. A key point in the approach taken in [20] is that the method restricts the dynamic range over which each coefficient is estimated by the use of, what is termed, 'frequency localising basis functions' (FLBFs) which span a desired frequency region. These functions allow the normal matrix to take on a near block diagonal form and hence improve its conditioning. We also note that this results in a reduction of the amount of correlation in the normal matrix.

In contrast to orthonormal basis functions, FLBFs are not exactly orthogonal for any standard input signal, however, are 'nearly orthogonal' for a wide range of inputs. Thus an exact property is traded for an approximate property with the aim of achieving some degree of robustness. This line of reasoning mirrors the usual trade-off that exists between performance (under some nominal conditions) versus robustness (under other conditions).

A further point to note, is that the filters utilised in the frequency localising basis functions are essentially bandpass filters and hence, relatively simple to implement.

Under some specific assumptions it was shown in [20] that the achieved condition number of the LS problem, when the frequency localising basis functions have been utilised, is independent of the dynamic range. One of the assumptions made is with respect to the model structure, i.e. it was established for a Moving Average (MA) model. In this paper we obtain a similar bound for an Output Error (OE) model structure.

The structure of the paper is as follows. In Section II we begin by formulating the problem and demonstrating how the Frequency Localising Basis Functions are utilised in the LS problem. Next we discuss our result which shows the independence of the LS condition number (for a particular case) when the estimator uses the proposed basis functions. In Section IV we consider higher order Frequency Localising Basis Functions, which can describe systems with sharper resonances and can improve numerical conditioning when the input frequencies are close to each other. Then, in Section V we illustrate with an example, utilising real data, the benefits obtained when employing these functions in the estimation of a large dynamic range system. We present conclusions in Section VI.

II. Problem Statement

In this section we outline the general problem and provide a description of the frequency localising basis functions and how they are applied to improve the properties of a LS estimator.

Consider a single-input-single-output linear continuous time system, with input $\{u(t)\}_{t\in\mathbb{R}}$ and output $\{y(t)\}_{t\in\mathbb{R}}$, defined by the strictly proper transfer function

$$G(s) := \frac{B(s)}{A(s)}, \tag{1}$$

where

$$\begin{aligned} A(s) &:= s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 \\ B(s) &:= b_m s^m + b_{m-1}s^{m-1} + \cdots + b_0 \end{aligned} \tag{2}$$

and $n, m \in \mathbb{N}$ $(n > m)$.

Figure 1 shows the relationship between the input u, the noise e (which is assumed to be zero mean Gaussian white noise) and the output y of the system $G(s)$.

Also consider the use of periodic excitation signals applied to the input for the purpose of obtaining a frequency response model of the system. Hence, we work in the frequency domain. Specifically, to estimate a parametric model of the system, let $\{u(t)\}_{t\in\mathbb{R}}$ be a sum of sine waves of unit amplitude and equal phase at frequencies $\omega_1, \ldots, \omega_N$ ($N \in \mathbb{N}$) (i.e., $U(j\omega_k) = 1$ for $k = 1, \ldots, N$), and consider a

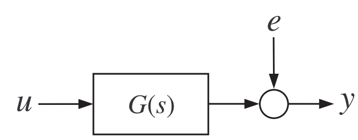

Fig. 1. Block diagram describing the relationship between the input u, the noise e and the output y of the system to be identified.

model given by the transfer function

$$\hat{G}(s) := \frac{\hat{B}(s)}{\hat{A}(s)}. \tag{3}$$

Here, $\hat{A}$ and $\hat{B}$ are polynomials which minimise the cost function

$$J := \sum_{k=1}^{N} \left| \frac{\hat{A}(j\omega_k)}{E(j\omega_k)} Y(j\omega_k) - \frac{\hat{B}(j\omega_k)}{E(j\omega_k)} U(j\omega_k) \right|^2 \tag{4}$$

where

$$\begin{aligned} \frac{\hat{A}(s)}{E(s)} &= 1 + \sum_{k=1}^{\tilde{n}} \alpha_k F_{2k-1}(s) \\ \frac{\hat{B}(s)}{E(s)} &= \sum_{k=1}^{\tilde{n}} \beta_k F_{2k}(s) \end{aligned} \tag{5}$$

and $\alpha_1, \ldots, \alpha_{\tilde{n}}, \beta_1, \ldots, \beta_{\tilde{n}} \in \mathbb{C}$ $(\tilde{n} \in \mathbb{N})$. The monic polynomial $E(s)$ is defined as the denominator of the right side of (5). The functions F_k, are those we term Frequency Localising Basis Functions [20], and are of the form

$$F_k(s) := s^{k-1} p_k \prod_{i=1}^{k} \frac{1}{s + p_i}; \qquad k = 1, \ldots, 2\tilde{n}, \tag{6}$$

where $0 < p_1 < \cdots < p_{2\tilde{n}} < \infty$.

To obtain the values of $\alpha_1, \ldots, \alpha_{\tilde{n}}, \beta_1, \ldots, \beta_{\tilde{n}}$, we can rewrite the problem of minimising J as a LS problem:

$$\hat{\theta} = (\mathbf{X}^H \mathbf{X})^{-1} \mathbf{X}^H \mathbf{Y} \tag{7}$$

where

$$\begin{aligned} \hat{\theta} &:= \begin{bmatrix} \alpha_1 & \beta_1 & \cdots & \alpha_{\tilde{n}} & \beta_{\tilde{n}} \end{bmatrix}^T \in \mathbb{C}^{2\tilde{n}\times 1} \\ \mathbf{X} &:= \begin{bmatrix} Y_1^1 & U_1^1 & \cdots & Y_1^{\tilde{n}} & U_1^{\tilde{n}} \\ \vdots & \vdots & & \vdots & \vdots \\ Y_N^1 & U_N^1 & \cdots & Y_N^{\tilde{n}} & U_N^{\tilde{n}} \end{bmatrix} \in \mathbb{C}^{N\times 2\tilde{n}} \\ \mathbf{Y} &:= \begin{bmatrix} Y(j\omega_1) & \cdots & Y(j\omega_N) \end{bmatrix}^T \in \mathbb{C}^{N\times 1} \\ Y_k^i &:= -F_{2i-1}(j\omega_k) Y(j\omega_k) \\ U_k^i &:= F_{2i}(j\omega_k) U(j\omega_k); \qquad k = 1, \ldots, N; \quad i = 1, \ldots, \tilde{n} \end{aligned} \tag{8}$$

and ^{H} is the complex conjugate transpose.

As in any LS problem the normal matrix, from (7), is defined as $(\mathbf{X}^H\mathbf{X})$. We note again that we do not advocate explicitly forming and inverting the normal matrix and suggest that techniques [7, 6] such as Cholesky Factorisation, Householder Transformation and QR Factorisation be utilised instead to obtain the LS solution.

To re-parameterise the model in terms of the coefficients $\hat{a}_1, \ldots, \hat{a}_n, \hat{b}_1, \ldots, \hat{b}_m \in \mathbb{R}$ $(n, m \in \mathbb{N})$ of $\hat{A}$ and $\hat{B}$ respectively, involves only a simple transformation [20], i.e. for the parameters of $\hat{A}$ let

$$\begin{aligned} M^k(s) &:= F_k(s)E(s) \\ &= m^k_{n-1}s^{n-1} + \cdots + m^k_{k-1}s^{k-1}; \quad k = 1, \ldots, n, \end{aligned} \tag{9}$$

where k represents the kth basis function, then

$$\mathbf{\hat{a}} = M\underline{\alpha} + \mathbf{e}, \tag{10}$$

where $\mathbf{\hat{a}}$ and $\mathbf{e}$ are the parameter vectors of $\hat{A}(s)$ and $E(s)$ respectively, $\underline{\alpha}$ is the vector of parameters in (5) and

$$M := \begin{bmatrix} m^1_0 & 0 & \cdots & 0 \\ m^1_1 & m^3_1 & \ddots & \vdots \\ \vdots & \vdots & & 0 \\ m^1_{n-1} & m^3_{n-1} & \cdots & m^{2\tilde{n}-1}_{n-1} \end{bmatrix}. \tag{11}$$

The re-parameterisation for $\hat{B}(s)$ follows similar lines.

III. Bound on the condition number

The condition number of the LS problem as described in Section II is defined as

$$\kappa := \frac{\lambda_{\max}(\mathbf{X}^H\mathbf{X})}{\lambda_{\min}(\mathbf{X}^H\mathbf{X})} = \frac{\sigma^2_{\max}(\mathbf{X})}{\sigma^2_{\min}(\mathbf{X})}, \tag{12}$$

where $\lambda_{\max}$ and $\lambda_{\min}$ are the maximum and minimum eigenvalues respectively and $\sigma_{\max}$ and $\sigma_{\min}$ are the maximum and minimum singular values respectively.

We now utilise this definition to show the existence of an upper bound on the condition number when Frequency Localising Basis Functions are used in the parametrisation of the estimated model. The following result is obtained:

Theorem 1: Consider the LS problem stated in Section II, and assume that:

1) $0 < G_{\min} \le G_{\max} < \infty$, where[1]

$$\begin{aligned} G_{\min} &:= \min\left\{1, \inf_{\omega\in\mathbb{R}} |G(j\omega)|\right\} \\ G_{\max} &:= \max\left\{1, \sup_{\omega\in\mathbb{R}} |G(j\omega)|\right\} \end{aligned} \tag{13}$$

2) The number of sine waves in $\{u(t)\}_{t\in\mathbb{R}}$, N, is equal to $2\tilde{n}$.
3) The frequencies of the sine waves in $\{u(t)\}_{t\in\mathbb{R}}$ are given by[2]

$$\begin{aligned} \omega_1 &= \gamma^{-1/2}p_1 \\ \omega_k &= \sqrt{p_{k-1}p_k}; \qquad k = 2, \ldots, 2\tilde{n}, \end{aligned} \tag{14}$$

and the breakpoints for the basis functions, p_k, are logarithmically spaced; i.e.,

$$p_{k+1} = \gamma p_k; \qquad k = 1, \ldots, 2\tilde{n} - 1 \tag{15}$$

for some $\gamma > 1$.

Then, if

$$\inf_{i\in\mathbb{N}} |F_i(j\omega_i)| > 4\frac{G_{\max}}{G_{\min}} \sum_{k=1}^{\infty} \sup_{i\in\mathbb{N}} |F_i(j\omega_{i+k})|, \tag{16}$$

there is a constant $K > 0$, independent of N, such that $\kappa \le K$.

Proof: According to (12), we need bounds for $\lambda_{\min}(\mathbf{X}^H\mathbf{X}) = \sigma^2_{\min}(\mathbf{X})$ and $\lambda_{\max}(\mathbf{X}^H\mathbf{X}) = \sigma^2_{\max}(\mathbf{X})$. To achieve this, we can use Wittmeyer's bounds on the eigenvalues of a matrix [21, 3]. For $\sigma_{\min}(\mathbf{X})$, by (16) and the fact that by (6), $|F_i(j\omega_{i+k})| \ge |F_i(j\omega_{i-k})|$ whenever $1 \le i - k < i + k \le 2\tilde{n}$, we have that

$$\begin{aligned} \sigma_{\min}(\mathbf{X}) &\ge \min_{i=1,\ldots,N} \left(|\mathbf{X}_{ii}| - \sum_{\substack{k=1 \\ k\neq i}}^{N} \left| \frac{\mathbf{X}_{ik} + \mathbf{X}_{ki}}{2} \right| \right) \\ &\qquad - \max_{i=1,\ldots,N} \sum_{k=1}^{N} \left| \frac{\mathbf{X}_{ik} - \mathbf{X}_{ki}}{2} \right| \\ &\ge G_{\min}\underline{h}_0 - 2G_{\max}\sum_{k=1}^{\infty}\bar{h}_k - 2G_{\max}\sum_{k=1}^{\infty}\bar{h}_k \\ &= G_{\min}\underline{h}_0 - 4G_{\max}\sum_{k=1}^{\infty}\bar{h}_k > 0, \end{aligned} \tag{17}$$

where

$$\begin{aligned} \underline{h}_0 &:= \inf_{i\in\mathbb{N}} |F_i(j\omega_i)| \\ \bar{h}_k &:= \sup_{i\in\mathbb{N}} |F_i(j\omega_{k+i})|; \qquad k \in \mathbb{N}_0 \,. \end{aligned} \tag{18}$$

Next, to bound $\sigma_{\max}(\mathbf{X})$ we have that

$$\begin{aligned} \sigma_{\max}(\mathbf{X}) &\le \max_{i=1,\ldots,N} \sum_{k=1}^{N} \left| \frac{\mathbf{X}_{ik} + \mathbf{X}_{ki}}{2} \right| \\ &\qquad + \max_{i=1,\ldots,N} \sum_{k=1}^{N} \left| \frac{\mathbf{X}_{ik} - \mathbf{X}_{ki}}{2} \right| \\ &\le 2G_{\max}\sum_{k=0}^{\infty}\bar{h}_k + 2G_{\max}\sum_{k=1}^{\infty}\bar{h}_k \\ &\le 4G_{\max}\sum_{k=0}^{\infty}\bar{h}_k < \infty \,. \end{aligned} \tag{19}$$

[1] Condition 1 is equivalent to assuming that G does not have any poles nor zeros on the imaginary axis.

[2] The value of ω_1 has been chosen so that the frequencies in $\{u(t)\}_{t\in\mathbb{R}}$ are logarithmically spaced (see Corollary 1 for the proof of this).

Thus,

$$\kappa = \frac{\sigma^2_{\max}(\mathbf{X})}{\sigma^2_{\min}(\mathbf{X})} \leq \frac{\left(4G_{\max}\sum_{k=0}^{\infty}\bar{h}_k\right)^2}{\left(G_{\min}\underline{h}_0 - 4G_{\max}\sum_{k=1}^{\infty}\bar{h}_k\right)^2} = 16\left(\frac{G_{\max}}{G_{\min}}\right)^2\left(\frac{\sum_{k=0}^{\infty}\bar{h}_k}{\underline{h}_0 - 4\frac{G_{\max}}{G_{\min}}\sum_{k=1}^{\infty}\bar{h}_k}\right)^2 =: K\ . \tag{20}$$

This concludes the proof. ■

Remark 1: Condition (16) depends on the ratio $G_{\max}/G_{\min}$. This quantity is always greater than or equal to 1, and the bound on the condition number of the LS problem is proportional to its square. Thus, it is important, for numerical reasons, to choose frequency points ω for which $|G(j\omega)|$ is neither too big nor too small. In particular, it is not convenient to choose points well beyond the cut-off frequency of the system, since for these points $|G(j\omega)|$ can be very small.

Remark 2: Theorem 1 establishes that when utilising Frequency Localising Basis Functions, the condition number is uniformly bounded irrespective of the system dynamic range.

We next establish a lower bound on the logarithmic spacing, γ, for which assumption 3 in Theorem 1 is satisfied and for which the condition (16) holds.

Corollary 1: Under the assumptions of Theorem 1, condition (16) holds if

$$\gamma > \left(2e^{15/448}\frac{G_{\max}}{G_{\min}} + \sqrt{4e^{15/224}\left(\frac{G_{\max}}{G_{\min}}\right)^2 + 1}\right)^2 . \tag{21}$$

Proof: By (15), we have that

$$p_k = \gamma^{k-1}p_1; \quad k = 1, \ldots, N \tag{22}$$

then, by (14),

$$\omega_1 = \gamma^{-1/2}p_1$$
$$\omega_k = \sqrt{\gamma^{k-2}p_1\gamma^{k-1}p_1} = \gamma^{k-3/2}p_1; \quad k = 2, \ldots, 2\tilde{n}\ . \tag{23}$$

Thus,

$$\begin{aligned}
&|F_i(j\omega_{i+k})| \\
&= |\omega_i^{i-1}p_i|\prod_{l=1}^{i}\frac{1}{|j\omega_{i+k}+p_l|} \\
&= |(\gamma^{i-3/2}p_1)^{i-1}\gamma^{i-1}p_1|\prod_{l=1}^{i}\frac{1}{|j\gamma^{i+k-3/2}p_1+\gamma^{l-1}p_1|} \\
&= \frac{\sqrt{\gamma^{1-2ik}}}{\prod_{l=1}^{i}\sqrt{1+\gamma^{-2(k+l)+3}}} \\
&= \frac{\gamma^{-k}}{\sqrt{1+\gamma^{-2(k+i)+3}}}|F_{i-1}(j\omega_{i+k-1})| \\
&\leq |F_{i-1}(j\omega_{i+k-1})|; \qquad i = 2, 3, \ldots,
\end{aligned} \tag{24}$$

hence

$$\underline{h}_0 = \lim_{i\to\infty}|F_i(j\omega_i)|$$
$$\bar{h}_k = |F_1(j\omega_{k+1})| = \sqrt{\frac{\gamma^{1-2k}}{1+\gamma^{1-2k}}}; \quad k \in \mathbb{N}_0\ . \tag{25}$$

Also,

$$|F_1(j\omega_1)| = \sqrt{\frac{\gamma}{1+\gamma}} < 1\ . \tag{26}$$

Now,

$$\bar{h}_k = \sqrt{\frac{\gamma^{1-2k}}{1+\gamma^{1-2k}}} \leq \sqrt{\gamma^{1-2k}} = \gamma^{1/2-k}; \quad k \in \mathbb{N}_0, \tag{27}$$

hence

$$\sum_{k=1}^{\infty}\bar{h}_k \leq \sum_{k=1}^{\infty}\gamma^{1/2-k} = \gamma^{-1/2}\sum_{k=0}^{\infty}\gamma^{-k} = \frac{\gamma^{1/2}}{\gamma-1}\ . \tag{28}$$

On the other hand,

$$\begin{aligned}
\prod_{l=1}^{\infty}\sqrt{1+\gamma^{3-2l}} &= \exp\left(\frac{1}{2}\sum_{l=0}^{\infty}\ln(1+\gamma^{1-2l})\right) \\
&= \exp\left(\frac{1}{2}\ln\gamma + \frac{1}{2}\sum_{l=0}^{\infty}\ln(1+\gamma^{-1-2l})\right) \\
&\leq \sqrt{\gamma}\exp\left(\frac{1}{2}\sum_{l=0}^{\infty}\gamma^{-1-2l}\right) \\
&= \sqrt{\gamma}\exp\left(\frac{1}{2}\frac{\gamma}{\gamma^2-1}\right),
\end{aligned} \tag{29}$$

hence

$$\underline{h}_0 = \frac{\sqrt{\gamma}}{\prod_{l=1}^{\infty}\sqrt{1+\gamma^{3-2l}}} \geq \exp\left(-\frac{1}{2}\frac{\gamma}{\gamma^2-1}\right). \tag{30}$$

Therefore, condition (16) is satisfied if

$$\exp\left(-\frac{1}{2}\frac{\gamma}{\gamma^2-1}\right) > 4\frac{G_{\max}}{G_{\min}}\frac{\gamma^{1/2}}{\gamma-1}\ . \tag{31}$$

This condition cannot be explicitly solved for γ, but it can be slightly relaxed to give a simpler expression for γ. To do this we impose an additional constraint on γ. For instance, if we force $\gamma > 15$, then

$$\exp\left(-\frac{1}{2}\frac{\gamma}{\gamma^2-1}\right) \geq \exp\left(-\frac{1}{2}\frac{15}{15^2-1}\right) = e^{-15/448}. \tag{32}$$

Thus, (31) holds if

$$e^{-15/448} > 4\frac{G_{\max}}{G_{\min}}\frac{\gamma^{1/2}}{\gamma-1},$$

which is equivalent to

$$\gamma > \left(2e^{15/448}\frac{G_{\max}}{G_{\min}} + \sqrt{4e^{15/224}\left(\frac{G_{\max}}{G_{\min}}\right)^2+1}\right)^2. \tag{33}$$

Notice that since $G_{\max}/G_{\min} \geq 1$, the right side of (33) is always greater than 19, so forcing $\gamma > 15$ in (32) is not too conservative. Moreover, lower values of γ give lower bounds for $\exp(-\gamma/(2(\gamma^2-1)))$, which give even higher values for the lower bound of γ in (33). ■

Remark 3: A better bound for γ may be achieved by solving the inequality

$$\exp\left(-\frac{1}{2}\frac{\gamma}{\gamma^2-1}\right) > 4\frac{G_{\max}}{G_{\min}}\frac{\gamma^{1/2}}{\gamma-1}. \tag{34}$$

For example, if $G_{\max}/G_{\min} = 1$, (33) gives $\gamma > 19.06$, while (34) gives $\gamma > 18.82$. This shows that both results are quite similar.

Remark 4: Even though it has been shown that a low condition number is a good sign of numerical stability, it is also true that a high condition number does not necessarily imply ill-conditioning. For example, if $\mathbf{X}^T\mathbf{X}$ has a diagonal structure, with some very high and also some very low diagonal elements, this usually does not give rise to any numerical problems, since the inverse of $\mathbf{X}^T\mathbf{X}$ is a diagonal matrix whose elements are the inverses of the respective diagonal elements of $\mathbf{X}^T\mathbf{X}$. See e.g. [6, pp. 28] for more details.

IV. An Extension to Higher Order Filters

In order to improve the numerical stability of the LS problem for lower values of γ than those satisfying Corollary 1, we can consider sharper basis functions than those proposed in [20]. An alternative is to use filters of the form:

$$F_k(s) := s^{q(k-1)}p_k^q\prod_{i=1}^{k}\frac{1}{(s+p_i)^q}; \quad k = 1,\ldots,2\tilde{n}, \tag{35}$$

where $q \in \mathbb{N}$ and $0 < p_1 < \cdots < p_{2\tilde{n}} < \infty$.

Figure 2 shows the magnitude of the frequency response of the basis functions (35) for $p_1 = 1$, $p_2 = 10$ and $p_3 = 100$, with $q = 1$ and $q = 2$.

For these sharper basis functions we have the following result, which is an extension of Corollary 1.

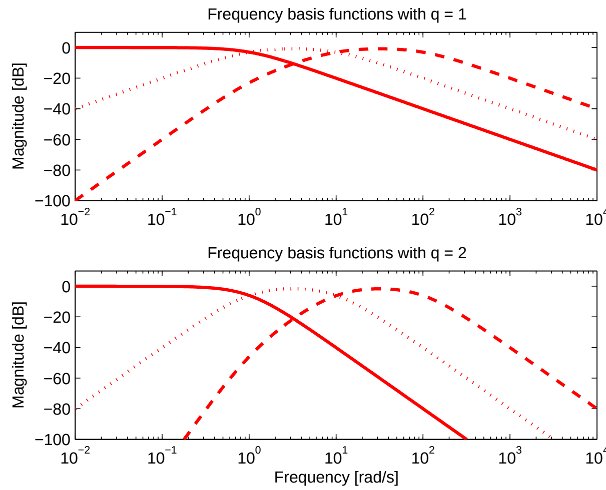

Fig. 2. Bode magnitude plots for basis functions for $q = 1$ (top) and $q = 2$ (bottom). Solid line: $F_1(s)$, dotted line: $F_2(s)$, dashed line: $F_3(s)$.

Corollary 2: Under the assumptions of Theorem 1, condition (16) holds for the basis functions given by (35) if

$$\exp\left(-\frac{q}{2}\frac{\gamma}{\gamma^2-1}\right) > 4\frac{G_{\max}}{G_{\min}}\frac{\gamma^{q/2}}{\gamma^q-1}. \tag{36}$$

Proof: The proof of Theorem 1 still holds if we replace (6) by (35). Thus, as in the proof of Corollary 1, we have that

$$\begin{aligned}\omega_1 &= \gamma^{-1/2}p_1\\ \omega_k &= \sqrt{\gamma^{k-2}p_1\gamma^{k-1}p_1} = \gamma^{k-3/2}p_1; \quad k = 2,\ldots,2\tilde{n}.\end{aligned} \tag{37}$$

Thus,

$$\begin{aligned}|F_i(j\omega_{i+k})| &= \left(\frac{\sqrt{\gamma^{1-2ik}}}{\prod_{l=1}^{i}\sqrt{1+\gamma^{-2(k+l)+3}}}\right)^q\\ &= \left(\frac{\gamma^{-k}}{\sqrt{1+\gamma^{-2(k+i)+3}}}\right)^q |F_{i-1}(j\omega_{i+k-1})|\\ &\leq |F_{i-1}(j\omega_{i+k-1})|; \quad i = 2,3,\ldots,\end{aligned} \tag{38}$$

hence

$$\begin{aligned}\underline{h}_0 &= \lim_{i\to\infty}|F_i(j\omega_i)|\\ \bar{h}_k &= |F_1(j\omega_{k+1})| = \left(\frac{\gamma^{1-2k}}{1+\gamma^{1-2k}}\right)^{q/2}; \quad k \in \mathbb{N}_0.\end{aligned} \tag{39}$$

Also,

$$|F_1(j\omega_1)| = \left(\frac{\gamma}{1+\gamma}\right)^{q/2} < 1. \tag{40}$$

Now,

$$\bar{h}_k = \left(\frac{\gamma^{1-2k}}{1+\gamma^{1-2k}}\right)^{q/2} \leq (\gamma^{1-2k})^{q/2} = \gamma^{q/2-qk}; \quad k \in \mathbb{N}_0, \tag{41}$$

so

$$\sum_{k=1}^{\infty} \bar{h}_k \leq \sum_{k=1}^{\infty} \gamma^{q/2-qk} = \gamma^{-q/2} \sum_{k=0}^{\infty} \gamma^{-qk} = \frac{\gamma^{q/2}}{\gamma^q - 1}. \quad (42)$$

Next,

$$\begin{aligned}
\prod_{l=1}^{\infty} (1+\gamma^{3-2l})^{q/2} &= \exp\left(\frac{q}{2}\sum_{l=0}^{\infty} \ln(1+\gamma^{1-2l})\right) \\
&= \exp\left(\frac{q}{2}\ln\gamma + \frac{q}{2}\sum_{l=0}^{\infty} \ln(1+\gamma^{-1-2l})\right) \\
&\leq \gamma^{q/2} \exp\left(\frac{q}{2}\sum_{l=0}^{\infty} \gamma^{-1-2l}\right) \quad (43) \\
&= \gamma^{q/2} \exp\left(\frac{q}{2}\frac{\gamma}{\gamma^2-1}\right),
\end{aligned}$$

hence

$$\begin{aligned}
\underline{h}_0 &= \lim_{i\to\infty} |F_i(j\omega_i)| \\
&= \left(\frac{\sqrt{\gamma}}{\prod_{l=1}^{i}\sqrt{1+\gamma^{3-2l}}}\right)^q \quad (44) \\
&\geq \exp\left(-\frac{q}{2}\frac{\gamma}{\gamma^2-1}\right).
\end{aligned}$$

Therefore, condition (16) is satisfied if

$$\exp\left(-\frac{q}{2}\frac{\gamma}{\gamma^2-1}\right) > 4\frac{G_{\max}}{G_{\min}}\frac{\gamma^{q/2}}{\gamma^q-1}. \quad (45)$$

■

Remark 5: Notice that the left side of (36) is a monotonically increasing function of γ, and its right side is monotonically decreasing in γ. Hence, if (36) holds for some $\gamma = \gamma_0$, it also holds for all $\gamma \geq \gamma_0$. Furthermore, since the left side of (45) increases with q, and its right side decreases with q, using higher values of q will improve the numerical conditioning of the LS problem.

V. Example

To illustrate the potential of the FLBFs we provide a practical example based on real experimental data[3] collected from a frequency response test of a large power transformer. It is well known that the fitting of a parametric model to this large dynamic range frequency response data using standard techniques is, at the very least, extremely difficult [18].

Figures 3 and 4 clearly show that the frequency localising basis functions provide an extremely good fit over 5 decades of frequency and 20 resonant modes. This is a very large dynamic range by any standard. Note that some low frequency data points were deliberately removed before fitting the model due to the possible contamination of the data with respect to the 50Hz interference from the mains power supply.

Figures 5 and 6 show the magnitude and phase responses of a model obtained by using the technique developed in [4], using orthogonal polynomials and 10 S-K iterations. As it can be seen, this technique gives a nice fit, except for some frequency regions. However, the coefficients of the polynomials obtained range from 1 to 9.4×10^{240}; this implies that there could be some potential overflow problems over larger frequency ranges.

The condition number obtained for this example, when using the FLBFs, is 1.9×10^{11}. This number seems extremely high, but as seen on Figures 3 and 4, there does not appear to be a numerical problem. This discrepancy motivates the study of a different measure for the conditioning of such problems for future work.

[3]The authors would like to thank Connell Wagner of Australia for making available the data.

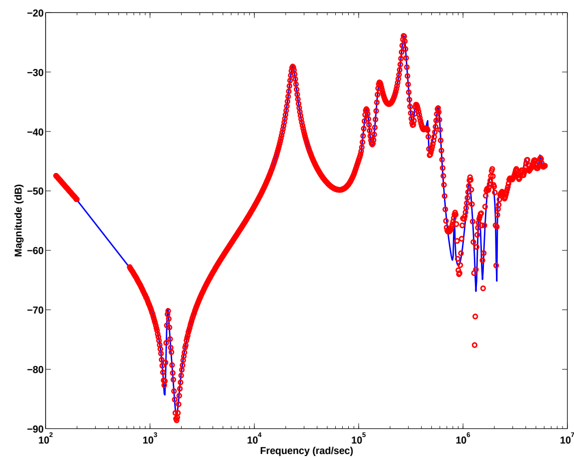

Fig. 3. Magnitude Response. Frequency response data (o), model (-).

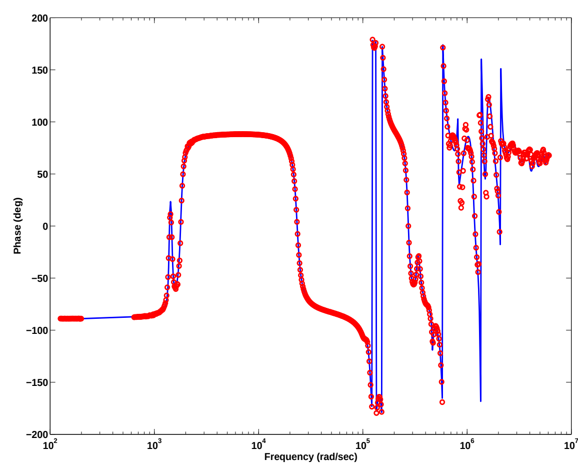

Fig. 4. Phase Response. Frequency response data (o), model (-).

VI. Conclusion

We have established a bound on the condition number, independent of the frequency range, of the least squares

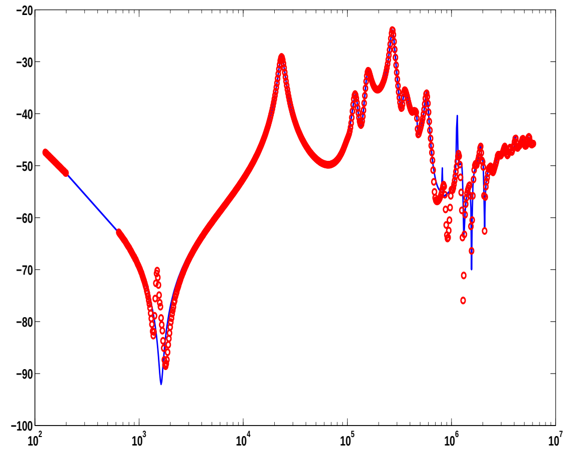

Fig. 5. Magnitude Response of model obtained using orthogonal polynomials [4]. Frequency response data (o), model (-).

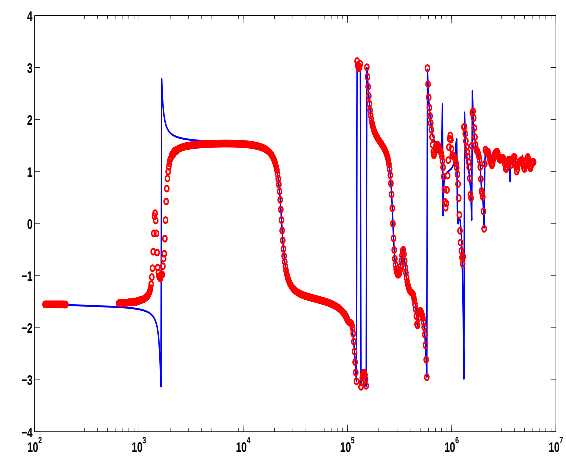

Fig. 6. Phase Response of model obtained using orthogonal polynomials [4]. Frequency response data (o), model (-).

problem for an Output Error model where frequency localising basis functions have been utilised. In addition, we have considered higher order basis functions, to model sharp resonances, and we have obtained explicit conditions on the frequency separation between these basis functions, under which the condition number can be bounded. It has also been demonstrated, via a real example, that frequency localising basis functions have merit when fitting a parametric model with a large dynamic range.

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