A Tutorial on Applications-Oriented Optimal Experiment Design

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Abstract Presented in this chapter is a tutorial on the design of input signals for system identification. The chapter concerns the details of basic optimal experiment design, computational issues, implementation and conditions that lead to an optimal input signal. It is also addressed how the two fundamental issues associated with system identification, model structure identification and model validation relate to experiment design. Throughout the chapter, a simple FIR example is employed to illuminate the concepts presented.

1 Introduction

The problem of how to determine an adequate model from experimental data – system identification – is a multi faceted topic, imposing quite high demands on the user. Two of the core issues are model structure selection and model validation. The key to simplify these problems and others lies in ensuring that suitable data is available for the modeling. What this means depends very much on what the model is going to be used for. We illustrate this with a simple example.

Example 1. Consider the finite impulse response (FIR) system

$$y_t = \sum_{k=1}^{n} \theta_k u_{t-k} + e_t \tag{1}$$

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where y_t is the output, u_t is the input and where $\{e_t\}$ is zero mean white noise with variance[1] $\lambda = \mathrm{E}[e_t^2]$, and where $\theta_k \in \mathbb{R}$, $k = 1, \ldots, n$ are the impulse response coefficients.

Let $\theta = \begin{bmatrix} \theta_1 & \ldots & \theta_n \end{bmatrix}^T \in \mathbb{R}^n$ and suppose that the objective is to estimate the frequency response $G(e^{j\omega}, \theta) = \sum_{k=1}^{n} \theta_k e^{-j\omega k}$ of the system. In particular let us assume that the frequency of interest is $\omega = 0$,, i.e. we are interested in the static gain $\sum_{k=1}^{n} \theta_k$ of the system. With a constant input $u_t = 1, t = 1, \ldots$, the system output will be

$$y_t = \left(\sum_{k=1}^{n} \theta_k \right) + e_t \qquad (2)$$

i.e. the output will fluctuate around the sought after quantity making the estimation problem very simple regardless of the order of the system. However, with this input the individual impulse response coefficients $\{\theta_k\}_{k=1}^{n}$ cannot be identified and thus, e.g., the frequency response at other frequencies than 0 can not be recovered from the data.   ∎

Thus, a given data set may be suitable for one application but not for another. In this chapter we will outline how one can formulate optimal experiment design problems whereby the application is taken into account.

The bibliography on optimal experiment design is extensive and include the by now classical texts [14, 37, 2, 29]. The survey papers [1, 28] provide good overviews of the area. Recently there have been contributions to design in the frequency domain [15], set-membership identification [6], applications-oriented experiment design [18, 23, 19, 4], plant-friendly design [9], convexification of the associated optimization problems [18, 23], robust experiment design [34], least-costly identification where the cheapest experiment that fulfills given performance specifications is designed [8], and closed loop experiment design [20]. Also the robustness properties of optimal experiment designs have been highlighted [26]. The four PhD theses [22, 25, 3, 32] provide good coverage of recent developments.

In this tutorial we will follow the framework outlined in [19]. We will start in the next section by discussing how to formulate such problems as optimization problems. Next we will consider how to "translate" such optimization problems into a computationally tractable format. This is rooted in recent developments of convex optimization. It is well known that typically solutions to this type of problem depend on properties of the unknown system. Practical ways to handle this are discussed in Section 4. For applications where the amount of properties that have to be extracted from the system by way of system identification is limited it turns out that applications-oriented optimal experiment designs provide "added value". The way the estimated model affects the application becomes less sensitive to design choices in the identification step. This is discussed in Section 5.

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[1] $\mathrm{E}[x]$ denotes expectation of a random variable x.

In order to not become submerged in technical details, we will keep the exposition simple by restricting ourselves to models of the type (1), estimated by way of standard least-squares.

2 Applications-oriented experiment design

2.1 The set of acceptable models

A simplified, but conceptually attractive way, to think of the objective of any modeling activity is that a model should be delivered to the user such that when it is used in the application, the resulting performance is acceptable. In fact, we could split the universe of all possible models into two sets: 1) $\mathscr{E}_{app}$: the set of models for which the performance of the application is acceptable, and 2) the set complementary to $\mathscr{E}_{app}$, consisting of all unacceptable models. The optimal experiment design problem thus becomes that of designing an experiment such that it is ensured that the resulting model belongs to the set of acceptable models $\mathscr{E}_{app}$. How to achieve this depends on the underlying assumptions about the data generating mechanism, i.e. the system, the used model and the estimation method. Here, for simplicity, we will consider standard least-squares estimation.

How to quantify what acceptable performance means is of course application dependent. Here we will for simplicity assume that this is captured by that a scalar function is less than some given value $1/\gamma$, we refer to Section 3.2 for details.

2.2 A stochastic framework

We will assume that data is generated by a system of the form (1). To simplify the discussion we will assume that the model structure is known, i.e. only the parameters $\theta = \begin{bmatrix} \theta_1 \ \ldots \ \theta_n \end{bmatrix}^T \in \mathbb{R}^n$ are unknown. We will return to the situation when the structure is unknown in Section 5. With $\hat{\theta}_N$ denoting the parameter estimate based on N input-output samples $\{y_t, u_t\}_{t=1}^N$, the source of the discrepancy between $\hat{\theta}_N$ and θ will then, neglecting unknown initial conditions, be due solely to the noise $\{e_t\}$ in (1). Assuming this quantity to be zero mean white noise implies that the error in the parameter estimate will be random. More specifically, the least-squares estimate of θ is given by [24]

$$\hat{\theta}_N := R_N^{-1} f_N \quad \text{where} \quad R_N := \sum_{t=1}^N \varphi_t \varphi_t^T, \quad f_N := \sum_{t=1}^N \varphi_t y_t$$

where $\varphi_t = \begin{bmatrix} u_{t-1} \ \ldots \ u_{t-n} \end{bmatrix}^T$. Above, we have assumed that R_N is full rank which corresponds to that data is persistently exciting [24]. Using (1), simple algebra gives

$$\hat{\theta}_N - \theta = R_N^{-1} \sum_{t=1}^{N} \varphi_t e_t$$

Assuming the noise to be normal distributed, this implies that also the parameter estimate is normal distributed[2]

$$\hat{\theta}_N - \theta \in N(0, \lambda R_N^{-1}) \tag{3}$$

where λ is the variance of the noise $\lambda = \mathrm{E}[e_t^2]$. From (3) it follows that $(\hat{\theta}_N - \theta)^T \frac{R_N}{\lambda}(\hat{\theta}_N - \theta)$ is χ^2-distributed with n degrees of freedom. With α denoting the 99.5% percentile of this distribution, this in-turn implies that the parameter estimate will end up in the confidence ellipsoid

$$\mathscr{E}_{id} := \left\{ \bar{\theta} : (\bar{\theta} - \theta)^T \frac{R_N}{\lambda}(\bar{\theta} - \theta) \leq \alpha \right\} \tag{4}$$

with 99.5% probability [24]. Notice that the ellipsoid $\mathscr{E}_{id}$ depends on the experimental conditions since R_N depends on the used input sequence.

2.3 Stochastic applications-oriented experiment design

With the set of acceptable models $\mathscr{E}_{app}$ given, one way of realizing the experiment design paradigm outlined in Section 2.1 is to design the input sequence such that

$$\mathscr{E}_{id} \subseteq \mathscr{E}_{app} \tag{5}$$

since then with at least 99.5% probability, the model parameter estimate will belong to $\mathscr{E}_{app}$.

The condition (5) can be achieved by many different types of experiments and in order to make the problem well posed some criterion has to be introduced. In [8] it was proposed that experiment design should try to minimize the experimental cost. Following this, one should try to achieve (5) by as "cheap" an experiment as possible. This leads to the following input signal design problem

$$\min_{u^N} V(u^N) \\ \text{s.t. } \mathscr{E}_{id} \subseteq \mathscr{E}_{app} \tag{6}$$

where $u^N = \begin{bmatrix} u_1, \ldots, u_N \end{bmatrix}^T$ and where the function $V(u^N)$ should measure the cost of the experiment.

To illustrate the approach we return to the frequency response estimation problem again.

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[2] $x \in N(m, P)$ denotes that the random variable x is normal distributed with mean m and covariance matrix P.

Example 2 (Example 1 cont'd). The frequency response can be expressed as $G(e^{j\omega}, \theta) = \theta^T \Gamma(e^{j\omega})$ where $\Gamma(e^{j\omega}) = \begin{bmatrix} e^{-j\omega} & \dots & e^{-j\omega n} \end{bmatrix}^T$, $\omega \in \mathbb{R}$. Suppose that we want the squared error

$$|G(e^{j\omega}, \hat{\theta}_N) - G(e^{j\omega}, \theta)|^2 = (\hat{\theta}_N - \theta)^T \Gamma(e^{j\omega}) \Gamma^*(e^{j\omega})(\hat{\theta}_N - \theta) \qquad (7)$$

(x^* denotes the complex conjugate transpose of x) to be less than $1/\gamma$ where the positive constant γ represents the desired accuracy. This means that

$$\mathscr{E}_{app} = \left\{ \bar{\theta} : (\bar{\theta} - \theta)^T \Gamma(e^{j\omega}) \Gamma^*(e^{j\omega})(\bar{\theta} - \theta) \leq \frac{1}{\gamma} \right\} \qquad (8)$$

∎

2.4 Alternative formulations

The formulation (6) was introduced in [19] but there exist other problem formulations that aim at achieving a similar objective. Here we briefly outline an approach that is closely connected to so called *L*-optimal design. We continue to use the frequency function estimation problem as example. The frequency function estimate is given by $G(e^{j\omega}, \hat{\theta}_N) = \hat{\theta}_N^T \Gamma(e^{j\omega})$, which, being linear in $\hat{\theta}_N$, has variance

$$\mathrm{E}\left[|G(e^{j\omega}, \hat{\theta}_N) - G(e^{j\omega}, \theta)|^2\right] = \Gamma^*(e^{j\omega}) R_N^{-1} \Gamma(e^{j\omega}) \qquad (9)$$

due to (3). Thus the constraint

$$\Gamma^*(e^{j\omega}) R_N^{-1} \Gamma(e^{j\omega}) \leq \beta \qquad (10)$$

for some suitably chosen constant β, captures that a certain quality of the estimate of $G(e^{j\omega}, \theta)$ is required. Replacing the set constraint in (6) by (10) gives

$$\begin{aligned} &\min_{u^N} V(u^N) \\ &\text{s.t. } \Gamma^*(e^{j\omega}) R_N^{-1} \Gamma(e^{j\omega}) \leq \beta \end{aligned} \qquad (11)$$

We refer to [23] for further details on this approach.

3 Computational aspects

When solving the input signal design problems (6) and (11) it is often more convenient to do it in the frequency domain. Below we will show that we can cast both these problems as convex programs in this domain.

3.1 Choice of decision variables

It is the matrix R_N in (6) and (11) that depends on the input. However, using the input sequence directly as decision variable leads to a non-convex problem. Turning to the frequency domain, the discrete Fourier transform (DFT) [27] of the input sequence is given by

$$U(e^{j\mu_k}) := \sum_{t=1}^{N} u_t e^{-j\mu_k t}, \quad \mu_k = 2\pi k/N, \quad k = 1, \dots, N \tag{12}$$

and with a periodic extension of the input, the DFT of the regressor sequence $\{\varphi_t\}$ is given by $\sum_{t=1}^{N} \varphi_t e^{-j\mu_k t} = \Gamma(e^{j\mu_k})U(e^{j\mu_k})$. By Parseval's theorem, [27], the matrix R_N can be expressed as

$$R_N = \sum_{t=1}^{N} \varphi_t \varphi_t^T = \frac{1}{N} \sum_{k=1}^{N} \Gamma(e^{j\mu_k})|U(e^{j\mu_k})|^2 \Gamma^*(e^{j\mu_k}) \tag{13}$$

Now we let the input be parameterized by the coefficients $\{c_t\}_{t=0}^{N-1}$ as

$$|U(e^{j\mu_k})|^2 = \sum_{t=0}^{N-1} c_t e^{-j\mu_k t} \tag{14}$$

Then the (p,q)-element of R_N is equal to $c_{|p-q|}$. Thus R_N is linearly parametrized in terms of these cofficients. As we will see below this is instrumental in order to convexify both (6) and (11).

Since $|U(e^{j\mu_k})|^2 \geq 0$, it has to hold that

$$\sum_{t=0}^{N-1} c_t e^{-j\mu_k t} \geq 0, \quad k = 1, \dots, N \tag{15}$$

for $\{c_t\}_{t=0}^{N-1}$ to be a valid parametrization. One implication of this is that $c_{N-j} = c_j$. These constraints thus have to be included.

3.2 Using the set of acceptable models

Let us now turn our attention to the set of acceptable models $\mathcal{E}_{app}$. Consider first the frequency function estimation example.

Example 3 (Example 2 cont'd). As we saw in (4) the confidence set $\mathcal{E}_{id}$ is an ellipsoid.

Furthermore, since (8) is quadratic in $\bar{\theta}$, $\mathcal{E}_{app}$ for the frequency response estimation problem also is an ellipsoid[3]. It is easy to verify, see [19], that the set constraint

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[3] To be more precise it is a degenerate ellipsoid since $\Gamma(e^{j\omega})\Gamma^*(e^{j\omega})$ is singular

$\mathscr{E}_{id} \subseteq \mathscr{E}_{app}$ is equivalent to

$$R_N - \gamma \lambda \alpha \Gamma(e^{j\omega}) \Gamma^*(e^{j\omega}) \geq 0 \qquad (16)$$

Now, since R_N is linear in the decision variables $\{c_t\}_{t=0}^{N-1}$, this inequality is a linear matrix inequality (LMI) which is a convex constraint. If $V(u^N)$ is the input energy, i.e. $V(u^N) = \sum_{t=1}^{N} u_t^2$ which in turn, using Parseval's theorem, can be written

$$V(u^N) = c_0 \qquad (17)$$

then the optimal input signal design problem (6) can be written

$$\min_{c_0,\dots,c_{N-1}} c_o \qquad (18)$$

$$\text{s.t.} \begin{bmatrix} c_o & c_1 & & \dots & c_{N-1} \\ c_1 & c_0 & c_1 & \dots & c_{N-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ c_{N-2} & c_{N-1} & \ddots & \ddots & c_1 \\ c_{N-1} & c_{N-2} & \dots & c_1 & c_0 \end{bmatrix} - \gamma \lambda \alpha \Gamma(e^{j\omega}) \Gamma^*(e^{j\omega}) \geq 0 \qquad (19)$$

$$\sum_{t=0}^{N-1} c_t e^{-j\mu_k t} \geq 0, \quad k = 1,\dots,N \qquad (20)$$

$$c_{N-j} = c_j, \quad j = 1,\dots,N-1 \qquad (21)$$

which is a semi-definite program and hence convex.

For example for $\omega = 0$, i.e. the static gain estimation problem discussed in Example 1, we have

$$\min_{c_0,\dots,c_{N-1}} c_o \qquad (22)$$

$$\text{s.t.} \begin{bmatrix} c_o & c_1 & & \dots & c_{N-1} \\ c_1 & c_0 & c_1 & \dots & c_{N-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ c_{N-2} & c_{N-1} & \ddots & \ddots & c_1 \\ c_{N-1} & c_{N-2} & \dots & c_1 & c_0 \end{bmatrix} - \gamma \lambda \alpha \begin{bmatrix} 1 & 1 & & \dots & 1 \\ 1 & 1 & \dots & & 1 \\ \vdots & \vdots & & & \vdots \\ 1 & 1 & \dots & & 1 \\ 1 & 1 & \dots & & 1 \end{bmatrix} \geq 0 \qquad (23)$$

$$\sum_{t=0}^{N-1} c_t e^{-j\mu_k t} \geq 0, \quad k = 1,\dots,N \qquad (24)$$

$$c_{N-j} = c_j, \quad j = 1,\dots,N-1 \qquad (25)$$

The diagonal elements in the matrix inequality give $c_0 \geq \gamma \lambda \alpha$. However, then it is straightforward to see that the choice $c_0 = c_1 = \dots = c_{N-1} = \gamma \lambda \alpha$ both satisfies the matrix inequality and the positivity constraints (24). This solution corresponds to

$$|U(e^{j\mu_k})|^2 = \begin{cases} N\gamma\lambda\alpha & k = N \\ 0 & k = 1,\ldots,N-1 \end{cases}$$

which in turn (taking the inverse DFT) corresponds to

$$u_t = \sqrt{\gamma\lambda\alpha}, \quad t = 1,\ldots,N$$

Thus we have proven that a constant input signal is optimal when the static gain is the quantity of interest. ∎

In general, $\mathcal{E}_{app}$ may not be convex. One can then use the following approximation. Consider a set of acceptable models given by

$$\mathcal{E}_{app} = \left\{ \bar{\theta} : V_{app}(\bar{\theta}) \leq \frac{1}{2\gamma} \right\} \tag{26}$$

for some function $V_{app}(\bar{\theta})$ that has a global minimum equal to zero at the true parameter value θ. A second order approximation is then given by

$$V_{app}(\bar{\theta}) \approx \frac{1}{2}(\bar{\theta} - \theta)^T V''_{app}(\theta)(\bar{\theta} - \theta) \tag{27}$$

since the gradient $V'_{app}(\theta)$ is zero by construction. Also, by construction, the second derivative $V''_{app}(\theta)$ is positive definite at the global optimum and when the approximation (27) is used in (26) the set of acceptable models is an ellipsoid given by

$$\mathcal{E}_{app} \approx \left\{ \bar{\theta} : (\bar{\theta} - \theta)^T V''_{app}(\theta)(\bar{\theta} - \theta) \leq \frac{1}{\gamma} \right\} \tag{28}$$

Following the reasoning in Example (3) gives that the condition $\mathcal{E}_{id} \subseteq \mathcal{E}_{app}$ (5) can be approximated by

$$R_N - \gamma\lambda\alpha V''_{app}(\theta) \geq 0 \tag{29}$$

3.3 Using variance constraints

Now we turn to the alternative constraint formulation (10) which involves the inverse of R_N. By making use of Schur complements, see e.g. [38], it can be translated to a constraint that is linear in R_N. Consider the partitioned hermitian matrix $M = \left(\begin{smallmatrix} A & B \\ B^* & C \end{smallmatrix} \right)$ where A is square and non-singular. The Schur complement of A in M is defined as $S_A := C - B^* A^{-1} B$ and it holds that $M \geq 0$ if and only if $A > 0$ and $S_A \geq 0$. The constraint (11) can be written as $\beta - \Gamma^*(e^{j\omega})R_N^{-1}\Gamma(e^{j\omega}) \geq 0$ where the left-hand side is the Schur complement of R_N in the partitioned matrix $M_R := \left(\begin{smallmatrix} R_N & \Gamma(e^{j\omega}) \\ \Gamma(e^{j\omega})^* & \beta \end{smallmatrix} \right)$. Since $R_N > 0$ by assumption it follows that (11) is equivalent to $M_R \geq 0$. Using the same arguments again we find that $M_R \geq 0$ if and only

if the Schur complement of β in M_R is positive semi-definite (β is positive by assumption):

$$R_N - \frac{1}{\beta}\Gamma(e^{j\omega})\Gamma^*(e^{j\omega}) \geq 0 \tag{30}$$

Note that, with $\beta = \frac{1}{\lambda\alpha\gamma}$, (30) is also equivalent to the constraint (29) with $V_{app}(\hat{\theta}_N)$ given by (7). Thus we end up with exactly the same optimization problem (18)–(21) as before, but with β replacing $\frac{1}{\lambda\alpha\gamma}$. We refer to [23] for generalizations.

4 How to handle system dependency of the optimal solution

In the formulation (6), typically either $\mathscr{E}_{id}$ or $\mathscr{E}_{app}$ (or both) depend on the true system, i.e., we have $\mathscr{E}_{id} = \mathscr{E}_{id}(\theta)$ and $\mathscr{E}_{app} = \mathscr{E}_{app}(\theta)$. This usually implies that the optimal input signal will be a function of something which is unknown prior to the experiment.

In our FIR example, $\mathscr{E}_{id}$ is independent of θ, since it is a linearly parameterized model structure (unlike, e.g., ARX or ARMAX structures, where the covariance does depend on θ). For the frequency estimation problem, $\mathscr{E}_{app}$ is also independent of θ. However, if we were interested instead in estimating the squared magnitude of the frequency function at a given frequency, $|G(e^{j\omega})|^2$, then a Taylor expansion shows that $\mathscr{E}_{app}$ is approximately

$$\mathscr{E}_{app}(\theta) = \left\{ \bar{\theta} : (|G(e^{j\omega}, \bar{\theta})|^2 - |G(e^{j\omega}, \theta)|^2)^2 \leq \frac{1}{\gamma} \right\}$$

$$\approx \left\{ \bar{\theta} : (\bar{\theta} - \theta)^T \Gamma(e^{j\omega})|G(e^{j\omega}, \theta)|^2 \Gamma^*(e^{j\omega})(\bar{\theta} - \theta) \leq \frac{1}{4\gamma} \right\}$$

where the dependence on θ through $|G(e^{j\omega}, \theta)|^2$ is clear.

The fact that the optimal input signal is a function of the true plant is not necessarily a problem if we already had some prior information, say from a previous experiment, and our goal is to design a new experiment to improve an existing model. In this case, we can simply replace in the formulation (6) $\mathscr{E}_{id}$ and $\mathscr{E}_{app}$ by $\mathscr{E}_{id}(\hat{\theta})$ and $\mathscr{E}_{app}(\hat{\theta})$, respectively, where $\hat{\theta}$ is a basic estimate of the plant; the input signal obtained in this way is called a *locally optimal input signal design* [13].

However, there are situations where we do not have enough reliable prior information to design our input signal based on a previous estimate of the plant. In the following subsections we describe two approaches to overcome this difficulty.

4.1 Robust designs

Even in the absence of a preliminary estimate of the plant, we might have some basic knowledge about its parameters. For example, from physical considerations it is sometimes possible to infer the order of magnitude of the dominant time constant, the static gain, etc. If this is the case, we can assume that we know a priori a set Θ such that $\theta \in \Theta$, and reformulate the optimal input signal design problem (6) to account for this information. One possibility is to consider the following robust version of (6):

$$
\min_{u^N} V(u^N)
$$
$$
\text{s.t. } \mathscr{E}_{id}(\theta) \subseteq \mathscr{E}_{app}(\theta), \text{ for all } \theta \in \Theta
$$
(31)

This problem corresponds to a *robust convex program*, which, save for a few exceptions, cannot be solved exactly in a computationally tractable way [7]. Here we present a simple approximate method to solve (31), based on a probabilistic relaxation technique known as the *scenario approach* [10]. For details, the reader is referred to [35]. The basic idea is to replace (31) by

$$
\min_{u^N} V(u^N)
$$
$$
\text{s.t. } \mathscr{E}_{id}(\theta_i) \subseteq \mathscr{E}_{app}(\theta_i), \quad i = 1, \ldots, m
$$
(32)

where $\theta_1, \ldots, \theta_m$ are independent random samples from a distribution P_θ in Θ. In [11] it was shown that if (32) is formulated in terms of d decision variables and m is chosen such that

$$
\binom{m}{d}(1-\varepsilon)^{m-d} < \delta
$$

for some constants $\delta, \varepsilon > 0$, then the solution of (32) will, with probability greater than $1 - \delta$, satisfy 'most' of the constraints in (31). More precisely, $P_\theta(\{\theta \in \Theta : \mathscr{E}_{id}(\theta) \nsubseteq \mathscr{E}_{app}(\theta)\}) < \varepsilon$ holds with probability $1 - \delta$.

In Corollary 1 in [10], the following more explicit (but conservative) expression for m was proposed: $m \geq (2/\varepsilon)\ln(1/\delta) + 2d + (2d/\varepsilon)\ln(2/\varepsilon)$. For example, for $d = 30$, $\varepsilon = 0.01$ and $\delta = 10^{-10}$, m should be at least 7864, which gives a tractable semidefinite program (solvable within 10 minutes on a computer with Intel Core 2 Duo CPU of 2.53 GHZ and a standard SDP solver)."

4.2 Adaptive and sequential designs

The idea of designing an experiment based on a previous estimate, and then redesigning the experiment from the previous one can be carried out even further, to

Fig. 1 Adaptive input signal
design.

arrive at the concept of adaptive input signal design. In this approach, a recursive estimator is employed to obtain a new estimate of the plant at each time instant, and the recursive estimate is then used to simultaneously redesign the input sequence. Figure 1 presents a block diagram of the adaptive technique.

As explained in [16], adaptive input signal design can achieve optimal performance (i.e., the same performance as if the experiment were designed based on the knowledge of the true system) under mild conditions, even though the convergence proofs are difficult and beyond the scope of the present tutorial. The interested reader is referred to [16, 17] for details, in particular the latter reference contains a convergence proof for adaptive signal design for ARX models.

5 Added value of optimal input signal design

In the previous sections we have focused on the formulation of applications-oriented optimal input signal design problems and how to solve the associated optimization problems. An obvious question is how much that can be gained by optimal input signal design. The answer to this question is clearly problem dependent, and also depends on what one compares with. The reduction in experiment time in the two case studies (a process control application and a mechanical system) presented in [5] was roughly a factor 4 as compared to a Pseudo-Random Binary Sequence. The paper [12] presents results from re-tuning of an MPC controller of a process having 34 inputs and 90 outputs at one of the world's largest refineries, the Hovensa refinery in Virgin Islands, United States. Partially through the use of optimal experiment design, the total modeling time was reduced by 90%.

By solving the optimal applications-oriented input signal design problem for different model complexities and different performance specifications it is possible to characterize how the minimal experimental cost depends on these quantities. This cost of complexity quantification, as it has become known, may provide valuable information for trading off performance in the application versus experimental effort in the identification problem. In Section 5.1 we discuss this for our standing example.

It is fairly obvious that optimal input signal design helps make system properties of importance easy to detect in the measured signals. However, a less obvious mechanism is that the optimal input signal tries to avoid to excite system behaviours that are inessential for the application. Thus it *hides* properties which are not important to the user. This dual aspect of the input signal has some very pleasant consequences, e.g.

1. It keeps the cost of the experiment at a reasonable level for models of high order.

2. It sometimes allows for consistent estimation of the properties of interest, even
   in the presence of unmodelled dynamics.

We will discuss this further in Section 5.2.

5.1 Cost of complexity

By focusing on specific properties of a given system, it is possible to reduce the
identification effort, measured in terms of the input signal energy required to esti-
mate the desired property within a given accuracy.

Example 4 (Example 3 cont'd). Suppose that (8) is modified to

$$\mathscr{E}_{app} = \left\{ \bar{\theta} : (\bar{\theta} - \theta)^T \Gamma(e^{j\omega})\Gamma^*(e^{j\omega})(\bar{\theta} - \theta) \leq \frac{1}{\gamma}, \ \forall \omega \in [0, \omega_B] \right\}, \qquad (33)$$

i.e. it is required that the accuracy of the frequency response is γ over the frequency
range $[0, \omega_B]$, where $0 < \omega_B \leq \pi$. Then, following [33], it can be shown that the
minimum input signal energy required (for large model orders n) is approximately

$$n\lambda\gamma\omega_B \qquad (34)$$

where λ is the variance of the noise e_t and $1/\gamma$ is the maximum allowed variance of
the frequency function estimate in the range $[0, \omega_B]$.

According to (34), the required input signal energy increases in proportion to the
model order n. However, this cost can be kept low by focusing on a more specific
frequency region, i.e., by reducing ω_B. This observation reenforces the statement
that input signal design can hide those properties which are unimportant, thus pro-
viding an energy efficient experiment.

As a comparison, it can be shown, see [33] for details, that using a PRBS (Pseudo
Random Binary Sequence) as input signal requires the energy $n\lambda\gamma\pi$ regardless of
the bandwidth ω_B to ensure that the parameter estimate ends up in (33). Thus, for
large model orders n, the reduction in required input energy through the use of
optimal input signal design can be significant for small to moderate bandwidths ω_B,
as compared to PRBS excitation.   ∎

5.2 Consistent estimation with unmodelled dynamics

A simple example of the phenomenon that sometimes optimal input signal design
allows the use of restricted models is given by the static gain estimation problem
studied in the examples of this chapter.

Example 5 (Example 4 cont'd). In Example 3 we saw that a constant input signal is
optimal for the problem of estimating the static gain. However, this particular input

signal has the additional property that it still allows for consistent estimation of the static gain if we use the following simple static model: $y_t = Ku_t$. The reason lies in the fact that a constant input signal completely hides all the other properties of the system by making them unidentifiable, This since, after the transient has died out, the system behaves as

$$y_t = \left(\sum_{k=1}^{n} \theta_k \right) u_t + e_t$$

when $u_t = u$ (constant).   ∎

A perhaps more interesting example is given in [25], where it has been shown that a nonminimum phase zero of a system G (of arbitrary order) at $z = z_o$ can be consistently estimated with a model of the form $y_t = b_1 u_{t-1} + b_2 u_{t-2}$ by using an input signal

$$u_t = \frac{1}{1 - z_o^{-1}} r_t$$

where r_t is zero mean white noise. This input signal depends on the property of interest, but it can be implemented using the adaptive scheme of Section 4.2 [31].

6 Concluding remarks

We have in this chapter provided the basics of applications-oriented experiment design. There exists a wide range of extensions. By viewing the input signal as a stationary signal with a continuous spectral density $\Phi_u(e^{j\mu})$, the spectral density takes the place of $|U(e^{j\mu_k})|^2$ in the formulation above and the discrete-time Fourier transform (DTFT) is used instead of the DFT. An input signal with the desired spectral density can be generated by filtering white noise through the spectral factor of $\Phi_u(z)$. The Kalman-Yakubovich-Popov lemma [36, 30] can be used to replace the positivity condition on the input spectrum by a linear matrix inequality. This allows a wide range of model structures to be used, e.g. ARX, ARMAX, output-error and Box-Jenkins models. We refer to [22] for details. Also closed loop experiment design can be handled in this way [20] and there is emerging work on applications-oriented experiment design for non-linear systems [21].

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